Lohse's metaphysical apparatus Gerling declares needless and useless. In the constructive part of his dissertation, Gerling dwells on the distinction between continuous and discrete quantity, admits the infinite divisibility of space and time, and constructs the infinite geometric progressions whose sums give respectively the distance and the time of running, before Achilles overtakes the tortoise. Gerling here repeats what Gregory St. Vincent had done long before, only Gerling uses letters, while Gregory assumed a special numerical case. Gregory is nowhere mentioned by Gerling. The sums of the two geometric progressions are values which in no way conflict with the estimate obtained from sensuous perception; Zeno's paradox, as interpreted by aid of the mathematical formulas, conflicts in no way with experience. Hence the puzzle is solved. Though a mathematician, Gerling does not feel the need of explaining the possibility of a variable reaching its limit.

As to the "Arrow" a sharp distinction between the continuous and the discrete is sufficient. In continuous quantity the number of possible subdivisions is *arbitrary*, and each subdivision is itself continuous. Hence Zeno's alleged denial of the infinite divisibility does not follow. Gerling treats the "Stade" with more than customary respect, and admits that, if one assumes with Zeno that space and time be not infinitely divisible, then it follows, as Zeno says, that half the time is equal to the whole time.

An entirely different type of discussion, more along the lines of Kant, profound and obscure, is given by Georg Wilhelm Friedrich Hegel. He holds the view that "Zeno's dialectic of matter has not been refuted to the present day; even now we have not got beyond it, and the matter is left in uncertainty."[1] He protects Aristotle against Bayle who objected to Aristotle's distinction between a potential and an actual subdivision of a line to infinity. Hegel keenly realizes the speculative importance of Zeno's paradoxes and points out that the dialectician of Elea had analyzed our concepts of time and space and had pointed out the contradictions involved therein; "Kant's antinomies do no more than Zeno did here."[2] Movement appears "in its distinction of pure self-identity and pure negativity, the point as distinguished from continuity."[3] This continuity is an absolute hanging together, an annihilation of all differences, of being by itself; the point on the other hand is pure existence by itself, the absolute distinctness from others, the suspension of all self-identity and all hanging together. In time and space the opposites are united in one, hence the contradiction as exhibited in motion. Hegel's position is a long way, still, from Georg Cantor's continuum, with its skilful union of continuity and discreteness. In the "dichotomy" the assumption of half a space is incorrect, says Hegel, "there is no half of space, for space is continuous; a piece of wood may be broken into two halves, but not space, and space only exists in movement."[4] Motion is connectivity, disintegration into an indefinite number of aggregates is its opposite.

[1] G. W. F. Hegel, *History of Philosophy*, transl. by E. S. Haldane, Vol. I, London, 1892, p. 265. See also Hegel's *Sämtl. Werke*, Bd. 13, 1833, pp. 314–327.

[2] Hegel [ed. Haldane], Vol. I, p. 277.

[3] Hegel, *op. cit.*, Vol. I, p. 268.

[4] Hegel, *op. cit.*, Vol. I, p. 271.

Somewhat more specific and comprehensible are the ideas set forth by his philosophical opponent, Johann Friedrich Herbart. Zeno's paradoxes are taken up by him in two works, his popular *Einleitung in die Philosophie* (1813) and his more technical and scientific *Allgemeine Metaphysik* (1828–9). Only in the latter work is the solution of the contradictions attempted. From it we quote:[1]

"The argument inevitably confuses those, who admit the infinite divisibility of the path and then console themselves with a corresponding infinite divisibility of the time, to such a degree that though at first willing to consider the process of dividing, which must continue to infinity, they soon in one leap consider the infinite number of time intervals as passed over, since they see that they must combine the infinite number of parts of the time as well as of the path to the place of overtaking, which they cannot do. The leap and the doubly infinite division are both faulty and amount to naught."

Thus, this infinite subdivision of the time and space is rejected by Herbart, because the mind is not able to *imagine* all the steps in the process. Imaginability is made the criterion of truth or error. This criterion throws out infinity at once; it throws out non-euclidean geometry and other parts of mathematics. We cannot really imagine things which we have never seen. Our senses are inaccurate, our intuitions are crude; hence it would seem to us impossible to build up sound mathematical theory, if everything unimaginable were to be cast aside. Herbart tries to explain motion by the concept of *velocity*, which seems itself to involve a contradiction, that Herbart endeavors to resolve by his theory of a "rigid line," a sort of continuum, which might have given rise to great possibilities upon more careful development. As it is, it offers greater obstacles by far than does the original "Dichotomy" or "Achilles" which it is intended to explain.

A still different attitude toward Zeno's paradoxes is taken by Friedrich Adolf Trendelenburg of the University of Berlin, in his *Logische Untersuchungen*, 1840, where he constructs his philosophic system upon the concept of motion. Constructive motion is common to the external world of being and to the internal world of thought, so that thought, as the counterpart of external motion, produces from itself space, time, and the categories. Motion is undefinable. In accordance with this view it is only through motion that Zeno's arguments against motion have come to be. For they depend upon the division of time and space, and the synthesis of those divisions. But division and synthesis are nothing but special forms of motion. What the proofs combat, they themselves use as the means of combat, and thereby testify to the controlling nature of motion. Trendelenburg and Kant evidently begin at opposite ends; Kant takes time and space as *a priori* ideas, and motion as secondary and dependent upon them; Trendelenburg makes motion the *a priori* idea, and pretends to derive time and space from it.

Friederich Ueberweg of the University of Königsberg[2] refers to our subject in different parts of his *Logik*. He says in one place that the "Achilles" proves too little; it proves merely that the tortoise cannot be overtaken within a definite series, and then claims that the tortoise cannot be overtaken anywhere and at

[1] J. T. Herbart, *Sämtl. Werke*, herausgeg. von Karl Kehrbach, Langensalza, Vol. VIII, 1893, p. 177.

[2] F. Ueberweg, *System der Logik*, 2. Aufl., p. 387 ff.

any time. True as this criticism may be, it does not illuminate the matter sufficiently to satisfy the reader.

Of the same type, but fuller in statement, is a criticism by Carl Prantl,[1] professor at the University of Munich. He claims that Zeno discarded the concept of continuity by considering only some particular points on a line and only some particular moments in time. By drawing his inferences from these disintegrated fragments of time and space, Zeno was able to advance contradictions in a picturesque manner. This conversion of the general and continuous into the particular and momentary will be encountered often, says Prantl, in those who care more for rhetorical form than for true philosophy.

Much confidence in his ability to clear up the mystery of Zeno's paradoxes is displayed by Eugen Carl Dühring in his *Kritische Geschichte der Philosophie*, first edition, 1869. Three concepts are necessary here: rest, motion and position. Usually only the first two are considered. At each moment (point) of time a moving body has a definite position but no motion. This fact makes it difficult to explain motion. He says further:[2] "The compelling force and real conclusiveness of the Eleatic contentions is to be found chiefly and almost exclusively in the logical necessity which does not permit the infinite to be thought of as completed, as enumerated so to speak, and concluded. . . . It is the concept of infinity which proves itself everywhere and also where it is not readily recognized, as the true cause of the contradictions." Dühring discusses infinity in several places of his works. He believes in the infinity usually set forth in the study of the calculus,—a variable which increases without limit, but at any moment has really a finite value. He makes war against the concept of an actual infinity—"jene wüste, sich widersprechende Unendlichkeit." "The infinite divisibility indicates . . . only this, that I can conceive the division of a quantity as far as I choose, without limit. If, on the other hand, I consider the division to infinity as really existing outside of my presentation of it, then there soon result the most manifold contradictions. . . . As regards motion, it must be recognized that it belongs to the empirical concepts, i. e., in our thinking there remains here always an unrecognizable residue, for we must give up the attempt to penetrate to the reasons of the phenomena." Georg Cantor criticizes Dühring in these words:

"The proofs of Dühring against the properly-infinite could be given in much fewer words and appear to me to amount to this, either that a definite finite number, however large it may be thought to be, can never be an infinite number, as follows immediately from the concept of it, or else that the variable, an unlimitedly large finite number, cannot be thought of with the quality of definiteness and therefore not with the quality of existence, as follows again from the nature of the variability. That not the least is hereby established against the conceivability of transfinite number, I feel certain; and yet, those proofs are taken as proofs against the reality of transfinite numbers. To me this mode of argumentation appears the same as if, from the existence of innumerable shades of green, we were to conclude that there can be no red."[3]

Dühring's explanation of infinity and of Zeno is accepted by Eduard Wellmann, in his historical monograph[4] of 1870. Another research, partly historical

[1] Carl Prantl, *Geschichte der Logik im Abendlande*, 1. Bd., Leipzig, 1855, pp. 10, 11.
[2] *Kritische Gesch. d. Philosophie*, Dr. E. Dühring, Leipzig, 1894, p. 49.
[3] Georg Cantor, *Grundlagen einer allg. Mannichfaltigkeitslehre*, Leipzig, 1883, p. 44.
[4] E. Wellmann, *op. cit.*, p. 23.

and partly expository, was published in 1867, by Johann Heinrich Loewe, a pupil of the philosopher, Anton Günther of Vienna. It is referred to by Knauer[1] as the most acute and satisfactory explanation that has yet been offered. "The solution of the riddle," says Loewe,[2] "appears to us to lie in the knowledge that contradictions must arise inevitably, as soon as space, time, and motion are considered at the same time from the stand-point of sensuous presentation and of non-sensuous conceptual reasoning." One point of view appeals to the imagination; the other to abstract thought. Sensuous perception can follow the process of infinite division only a little way, everything beyond is a matter of pure reason. Gerling's presentation of "Achilles" is an appeal to reason. As long as one considers the infinite multiplicity of small distances and of time-intervals, one approaches the riddles from the standpoint of abstract thought; when one appeals to the imagination, then the finite time and the finite length of the race stand out. Loewe seems still to hold to the old view that thought can recognize no end to a motion which extends over an infinite process. Hence the contradiction must stand, the antinomy is evident.

Thus we see that German philosophy down to the last quarter of the nineteenth century continually accentuates the existence of contradictions in the problem of motion.

Some English thinkers of the nineteenth century, who were interested in Zeno's arguments, came under the influence of Kantian philosophy. The Kantian attitude toward Zeno is described in the article "Zeno" in the eighth edition of the *Encyclopaedia Britannica* (1860) thus:

"He brought a most powerful mind to his task, and, curious to say, subsequent thinkers have very generally agreed in misunderstanding both his reasoning and his method, and it is only of late years that Kant, in his *Antinomies of the Pure Reason* (see *Kritik der Reinen Vernunft*) seized upon the much maligned doctrines of the Eleatic, and held them up to the admiration of all true thinkers as rare examples of acute and just thought. Bayle, in a clever paper on Zeno, in his *Dictionnaire*, makes him, according to custom, a sceptic. Brucker finds that Zeno surpasses his intelligence, and he is content to make him a pantheist. Others again, have charged him with nihilism. Zeno, fortunately, can afford to sit quite easy to all those affronts offered to his reason . . . they [arguments against motion] all take their rise, as Kant and Hamilton (*Lectures on Metaphysics*) have shown, from the inability of the mind to conceive either the ultimate indivisibility, or the endless divisibility, of space and time, as extensive and as protensive quantities. The possibility of motion, however certain as an observed fact, is thus shown to be inconceivable. To have discovered this peculiarity of our mental constitution, and to have stated it with eminent clearness, belongs to Zeno the Eleatic, and to him alone."

Sir William Hamilton puts this matter thus:[3]

"Time is a protensive quantity, and, consequently, any part of it, however small, cannot, without a contradiction, be imagined as not divisible into parts, and these parts into others *ad infinitum*. But the opposite alternative is equally impossible; we cannot think this infinite division. One is necessarily true; but neither can be conceived possible. It is on this inability of the mind to conceive either the ultimate indivisibility, or the endless divisibility of space and time, that the arguments of the Eleatic Zeno against the possibility of motion are founded,—arguments which at least show, that motion, however certain as a fact, cannot be conceived pos-

[1] Vincenz Knauer, *Die Hauptprobleme der Philosophie*, Wien u. Leipzig, 1892, p. 54.

[2] J. H. Loewe, *op. cit.*, p. 32.

[3] *Lectures on Metaphysics and Logic*, by Sir William Hamilton, Vol. I, Boston, 1863, Lecture 38, p. 530.

sible, as it involves a contradiction. . . . Now the law of mind, that the conceivable is in every relation bounded by the inconceivable, I call the Law of the Conditioned."

John Stuart Mill, in his *Logic*,[1] refers to Thomas Brown who considered the "Achilles" insoluble, and then offers a solution to the invention of which he lays no claim. It presents no new points of view. Herbert Spencer discusses questions of time and space in his *First Principles*[2] and concludes in general that "ultimate scientific ideas, then, are all representative of realities that cannot be comprehended." In particular, "halve and again halve the rate of movement for ever, yet movement still exists; and the smallest movement is separated by an impassable gap from no movement."

It is readily seen that the nineteenth century philosophers had penetrated deeper than most of their predecessors and had encountered difficulties previously neglected by Hobbes and others who seemed to think that they had solved the "Achilles" paradox by the mere statement that time, as well as space, was infinitely divisible. What came to be thoroughly realized since the time of Kant was the impossibility of *imagining* the "Achilles" from the standpoint of infinite divisibility of a distance, that all appeals to intuition were futile. When Spencer says that infinite divisibility cannot be "comprehended," and Thomas Brown and Sir William Hamilton say that motion is "insoluble" and "inconceivable," I take it that they mean simply that these processes are *unimaginable*, that they are beyond the reach of our sensual intuitions. I do not interpret them to mean that these processes are beyond the reach of logic, beyond the reach of the reasoning faculty so as to be, and forever remain, wholly mysterious. Mathematics includes among its results numerous teachings which one cannot "imagine." Probably no one claims to be truly able to visualize to himself the non-euclidean geometries; analysts do not claim to be able to imagine or see a continuous curve which has no tangent line at any of its points. Yet no modern mathematician rejects non-euclidean geometries and non-differentiable continuous curves.

These unimaginable mathematical creations are admitted into the science as a matter of necessity. Felix Klein states the issue as follows: "As the subjects of abstract geometry cannot be sharply comprehended through space intuition, one cannot rest a rigorous proof in abstract geometry upon mere intuition, but must go back to a logical deduction from axioms assumed to be exact."[3]

It so happens that England's two famous opium eaters, Thomas De Quincey and Samuel Taylor Coleridge, were interested in the "Achilles." Coleridge's critical powers were set forth by De Quincey in the following terms:[4]

"I had remarked to him that the sophism, as it is usually called, but the difficulty, as it should be called, of Achilles and the Tortoise, which had puzzled all the sages of Greece, was, in fact, merely another form of the perplexity which besets decimal fractions; that, for example, if

[1] *A System of Logic*, Vol. II, London, 1851, p. 381.
[2] H. Spencer, *First Principles of a New System of Philosophy*, New York, 1882, pp. 47–67.
[3] F. Klein, *Anwendung der Differential- und Integralrechnung auf Geometrie*. Leipzig, 1907, p. 19.
[4] *Tait's Magazine*, Sept. 1834, p. 514.

you throw ⅔ into a decimal form, it will never terminate, but be .666666, etc., *ad infinitum*. 'Yes,' Coleridge replied, 'the apparent absurdity in the Grecian problem arises thus,—because it assumes the infinite divisibility of space, but drops out of view the corresponding infinity of time.' There was a flash of lightning, which illuminated a darkness that had existed for twenty-three centuries."

As a matter of fact, Aristotle had seen that far. But Coleridge proceeded somewhat farther in an essay on Greek sophists in *The Friend*,[1] where he says:

"The few remains of Zeno the Eleatic, his paradoxes against the reality of motion, are mere identical propositions spun out into a sort of whimsical conundrums, as in the celebrated paradox entitled Achilles and the Tortoise, the whole plausibility of which rests on the trick of assuming a *minimum* of time while no *minimum* is allowed to space, joined with that of exacting from *intelligibilia*, *νοήμενα*, the conditions peculiar to objects of the senses *φαινόμενα* or *αἰνθανόμενα*."

What belongs to Coleridge himself in this passage is the contention that the sophism consists in applying to an idea conditions only properly applicable to sensuous *phaenomena*. Coleridge's argument was elaborated many years later in dialogue form, by Shadworth H. Hodgson. We give the critical part of the discussion:[2]

". . . being infinitely *divisible* is not the same thing as being infinitely *divided*. Actually to divide to infinity that hundredth part of a minute, in which (*phenomenally* as you say) Achilles overtakes the tortoise, is an infinitely long operation. . . . And *this* division you call upon Achilles to perform, before the tortoise can be overtaken, and to perform *phenomenally*. . . . You require that Achilles shall exhibit to the senses the infinite divisibility of time and space, which appertains to them truly indeed, but only as objects of imagination and thought. . . . The world of *thought* and *reality* is not a world apart, but is identical with the phenomenal world, only differently treated. . . . Neither is there any contradiction between them. Phenomenal motion is as infinitely divisible in *thought* as time and space are."

This explanation does not explain. Even as "objects of the imagination" the infinite divisibility of time and space is a source of perplexity. Our imagination is unable to follow Achilles to the end, through the infinities of time and space intervals. Moreover, "*thought* and *reality*" are indeed worlds "apart" whenever the time intervals, corresponding to the space-intervals passed over by Achilles, are so taken that they form together an infinite series that is *divergent*, so that, in thought, Achilles never overtakes the tortoise; in Zeno's traditoinal argument, "thought and reality" were "apart."

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[1] *Complete Works of S. T. Coleridge*, Vol. II, New York, 1856, p. 399.
[2] *Mind*, London, Vol. V, 1880, pp. 386–388.

[The remaining parts of this series are: *D.* VIEWED IN THE LIGHT OF AN IDEALISTIC CONTINUUM (G. Cantor); *E.* POST-CANTORIAN DISSENSION.]

LINEAR MOMENTUM, KINETIC ENERGY, AND ANGULAR MOMENTUM.

By E. B. WILSON, Massachusetts Institute of Technology.

The MONTHLY for December, 1914, contains an interesting and important note by E. V. Huntington on uniplanar rigid motion in which it is pointed out that the relation between angular acceleration and moments of forces is incorrectly given in a number of texts and in which the correct relation is clearly exhibited. We may point out that the theorem on energy is also at fault in some books. For instance, in Smith and Longley's *Theoretical Mechanics*, page 234, we find that the kinetic energy of a lamina moving in its own plane is the sum of two terms:

$$\text{K. E.} = \tfrac{1}{2}Mv_0^2 + \tfrac{1}{2}I_0\omega^2, \qquad \text{(A)}$$

where v_0 is the speed of any point of the lamina, I_0 the moment of inertia of the lamina about that point, and ω the angular velocity which is the same about all points.

This remarkably untrue result is of course "proved," not merely stated. The error is more grievous than that on the angular acceleration, because to err in a first order effect (a velocity) is far less excusable than to err in a second order effect (an acceleration). The knowledge that plane motion is *kinematically* a composite of rotational velocity about any point and a translational velocity of that point is so ingrained in the minds of some persons that they do not recognize that *dynamically* the composition is invalid. It is not that the right results are any harder to prove than the wrong ones; it is merely evidence in high quarters of the same sort of intellectual inattention that leads the sophomore or junior to "solve" all problems in rectilinear motion by his familiar formulas for uniform motion, however much the motion may be irregular.

We propose to give, in as direct a manner as possible, a correct treatment of linear momentum, kinetic energy, and angular momentum, whether they be measured absolutely or relative to a moving origin. (There will be no reference to moving *rotating* axes, which introduce comparatively difficult and delicate considerations.) We shall use a minimum amount of vector analysis, following the notations of Gibbs, to avoid writing the equations for each component separately. Let us denote by

$\boldsymbol{r}$, absolute position of any point P,	$\boldsymbol{v}$, absolute velocity of P,
$\boldsymbol{s}$, position of moving origin O,	$\boldsymbol{u}$, velocity of moving origin O,
$\boldsymbol{r}'$, position of P relative to O,	$\boldsymbol{v}'$, velocity of P relative to O.

(Absolute position and velocity mean position and velocity referred to a fixed origin; we mean no metaphysics.) We then have the equations

$$\boldsymbol{r} = \boldsymbol{s} + \boldsymbol{r}', \quad \boldsymbol{v} = \boldsymbol{u} + \boldsymbol{v}'. \qquad (1)$$

The second is the "parallelogram law" for compounding velocities; the first states that one vector side of a triangle is the sum of the other two sides as vectors.

Linear Momentum. Let particles of mass m_i $(i = 1, 2, \cdots)$ be situated at the points P_i. By definition the linear momenta of each individual mass and of the whole set of masses are respectively

$$m_i\boldsymbol{v}_i \quad \text{and} \quad \text{L. M.} = \sum_i m_i\boldsymbol{v}_i. \tag{2}$$

To simplify the second of these statements we introduce an ideal point G, called the center of mass or the center of gravity, by the definition

$$(\sum_i m_i)\bar{\boldsymbol{r}} = \sum_i m_i\boldsymbol{r}_i \quad \text{or} \quad M\bar{\boldsymbol{r}} = \sum_i m_i\boldsymbol{r}_i, \tag{3}$$

where $\bar{\boldsymbol{r}}$ is the absolute position of G, and M is the total mass of the system. If we differentiate with respect to the time and denote by $\bar{\boldsymbol{v}}$ the velocity of G, we have

$$M\bar{\boldsymbol{v}} = \sum_i m_i\boldsymbol{v}_i = \text{L. M.} \tag{4}$$

We may therefore state

THEOREM 1. *The linear momentum of a system of particles is equal to the product of the mass of the system and the velocity of its center of gravity.*

By substitution of (1) in (2) and reduction we have

$$\text{L. M.} = \sum_i m_i\boldsymbol{v}_i = \sum_i m_i\boldsymbol{u} + \sum_i m_i\boldsymbol{v}_i' = M\boldsymbol{u} + \sum_i m_i\boldsymbol{v}_i'.$$

Now $\Sigma m_i\boldsymbol{v}_i'$ is the relative linear momentum, by definition. Hence

THEOREM 2. *The linear momentum of a system is the sum of the relative linear momentum and of the momentum of a mass equal to the total mass of the system moving with the point of reference.*

THEOREM 3. *The relative linear momentum $\Sigma m_i\boldsymbol{v}_i'$ is equal to $M\bar{\boldsymbol{v}}'$, the product of the mass and the relative velocity of the center of gravity.*

The proof of Theorem 3 is left to the reader.

Kinetic Energy. By definition, the kinetic energies of each mass and of the whole set of masses are respectively

$$\tfrac{1}{2}m_i\boldsymbol{v}_i^2 \quad \text{and} \quad \text{K. E.} = \tfrac{1}{2}\sum_i m_i\boldsymbol{v}_i^2, \tag{5}$$

it being agreed that by the square of a velocity we mean the square of its magnitude, or, what amounts to the same thing, the scalar product of the vector velocity into itself,—thus $\boldsymbol{v}^2 = \boldsymbol{v}\cdot\boldsymbol{v}$.[1]

On substitution of (1) in (5) and reduction we find

$$\text{K. E.} = \tfrac{1}{2}\sum_i m_i\boldsymbol{v}_i\cdot\boldsymbol{v}_i = \tfrac{1}{2}\sum_i m_i(\boldsymbol{u} + \boldsymbol{v}_i')\cdot(\boldsymbol{u} + \boldsymbol{v}_i')$$

[1] The object of introducing the scalar product is to render the square of a vector which may be written as the sum of two vectors amenable to the ordinary distributive laws of multiplication; we may thereby avoid the use of the law of cosines.

$$= \tfrac{1}{2}\sum_i m_i \boldsymbol{u}\cdot\boldsymbol{u} + \sum_i m_i \boldsymbol{u}\cdot\boldsymbol{v}_i' + \tfrac{1}{2}\sum_i m_i \boldsymbol{v}_i'^2.$$

Or

$$\text{K. E.} = \tfrac{1}{2}M\boldsymbol{u}^2 + \boldsymbol{u}\cdot(M\bar{\boldsymbol{v}}') + \tfrac{1}{2}\sum_i m_i \boldsymbol{v}_i'^2, \tag{6}$$

when by Theorem 3 we have replaced $\Sigma m_i \boldsymbol{v}_i'$ by $M\bar{\boldsymbol{v}}'$.

THEOREM 4. *The kinetic energy of a system is the sum of* THREE *terms:* (*a*) *the relative kinetic energy* $\frac{1}{2}\Sigma m_i \boldsymbol{v}_i'^2$, (*b*) *the kinetic energy of the whole mass located at the moving point of reference* $\frac{1}{2}M\boldsymbol{u}^2$, (*c*) *the product of the mass M by the scalar product* $\boldsymbol{u} \circ \bar{\boldsymbol{v}}'$ *of the velocity of the point of reference and the relative velocity of the center of gravity.*

The third term (*c*), which is $M\boldsymbol{u}\cdot\bar{\boldsymbol{v}}'$, vanishes only when: ($\alpha$) the point of reference O is (at least momentarily) at rest, or (β) the center of gravity is (at least momentarily) at rest relative to O, or (γ) the velocity of the point O and the relative velocity of the center of gravity are perpendicular.

To compute the kinetic energy of a rigid lamina we *do not need* to use Theorem 4; we may apply the original definition (5). For if the lamina has a motion of pure translation (at the instant considered), we find by direct summation that K. E. $= \frac{1}{2}M\boldsymbol{v}^2$, where $\boldsymbol{v}$ is the velocity of any point; whereas if the lamina is rotating, the velocity of any point is $r\omega$ in magnitude, r being the distance from the instantaneous center to the point, and we again can sum directly to find K. E. $= \frac{1}{2}I_C\omega^2$, where $I_C = \Sigma m_i r_i^2$ is the moment of inertia with respect to the instantaneous center.

In many cases it is *desirable* to express the kinetic energy of the lamina with respect to a point O fixed in the lamina. Then Theorem 4 is applicable, and the term (*a*) may be simplified by using $r'\omega$ as the magnitude of $\boldsymbol{v}'$, the velocity relative to the fixed point. We have therefore

$$\text{K. E.} = \tfrac{1}{2}M\boldsymbol{u}^2 + M\boldsymbol{u}\cdot\bar{\boldsymbol{v}}' + \tfrac{1}{2}I'\omega^2, \tag{6'}$$

where $I' = \Sigma m_i r_i'^2$ is the moment of inertia relative to the point O. Moreover, if θ be the angle between the directions $\boldsymbol{u}$ and $\bar{\boldsymbol{r}}'$, the angle between $\boldsymbol{u}$ and $\bar{\boldsymbol{v}}'$ is $90° + \theta$, and as the magnitude of $\bar{\boldsymbol{v}}'$ is $r'\omega$, we have

$$\text{K. E.} = \tfrac{1}{2}M\boldsymbol{u}^2 - Mu\omega\bar{r}'\sin\theta + \tfrac{1}{2}I'\omega^2. \tag{6''}$$

The kinetic energy consists of *three* terms and does *not* reduce to $\frac{1}{2}M\boldsymbol{u}^2 + \frac{1}{2}I'\omega^2$ unless $u = 0$, $\omega = 0$, $\bar{\boldsymbol{r}}' = 0$, or $\sin\theta = 0$.

THEOREM 5. *The proposition* (*A*), *that the kinetic energy of a lamina is the sum of two terms* $\frac{1}{2}M\boldsymbol{v}_O^2 + \frac{1}{2}I_O\omega^2$, *representing the kinetic energy of translation and of rotation with reference to some point O, as if kinematic resolution were dynamically right, is true for all points O upon the circle which at the instant considered contains the instantaneous center and the center of gravity as diametrically opposite points, and for no points not on this circle except when there is no rotation and the construction breaks down.*

This pretty little geometric result is probably not new, but I have no re-

membrance of having seen it. The proof, which follows immediately from our work just above, will be left to the reader.

Example. Let a hoop of mass m, uniformly distributed, and of radius a be loaded at one point of the rim with a mass m'. Let the angular velocity be ω, and the inclination of the radius from the center to m' be φ (measured backward from the downward vertical). Find the kinetic energy.

We may solve in three different ways: (α) by using the instantaneous center, (β) by taking as point of reference the center of the hoop, (γ) by taking as point of reference the center of gravity. The details of the work will be left to the reader, who will find that the result obtained from (β) will check with the others only when three terms are used in the expression for the kinetic energy.

The following are well-known propositions of kinematics in the case of rigid bodies in space: (*a*) motion about a fixed point is at each instant rotation about some axis through that point, (*b*) motion in general may be regarded as composed of translation of an arbitrary point fixed in the body and rotation about some axis through that point, (*c*) for all the points of the body the directions of the axes are the same, (*d*) there is one of these axes such that the translation of every point on it is along the axis. It is by this suite of theorems that we prove that kinematically the instantaneous motion of a rigid body is screw motion.

Formula (6′) for the kinetic energy of a lamina is equally true for a rigid body-provided I' denotes the moment of inertia about the instantaneous axis of rotation. No new proof is needed. The term $M\boldsymbol{u}\cdot\bar{\boldsymbol{v}}'$, which must be added to the sum of the translatory and rotatory energies, will in general not vanish. It will vanish if the point of reference is the center of gravity or any point of the axis of the instantaneous screw (for then $\bar{\boldsymbol{v}}'$ and $\boldsymbol{u}$ are perpendicular). Moreover, if $\boldsymbol{u}\cdot\bar{\boldsymbol{v}}'$ vanishes for any point of the body, it will vanish for all points of the line through that point parallel to the axis of rotation (for $\boldsymbol{u}$ and $\bar{\boldsymbol{v}}'$ do not vary from point to point of this line). Hence from Theorem 5 we may obtain

THEOREM 6. *The kinetic energy of a rigid body in space may be regarded as a sum of the translatory and rotatory energies when and only when the point of reference (fixed in the body) lies on a right circular cylinder passing through the axis of the instantaneous screw and having the center of gravity of the body upon the element diametrically opposite to that axis.*

Angular Momentum. By definition, the angular momenta of each mass and of the whole set of masses about a given fixed point F are respectively

$$m_i\boldsymbol{r}_i \times \boldsymbol{v}_i \quad \text{and} \quad \text{A. M.} = \sum_i m_i\boldsymbol{r}_i \times \boldsymbol{v}_i, \tag{7}$$

where $\boldsymbol{r}$ is the radius from the point F and $\boldsymbol{r} \times \boldsymbol{v}$ denotes the vector product of $\boldsymbol{r}$ into $\boldsymbol{v}$.[1] The angular momentum is therefore a vector magnitude, since $\boldsymbol{r}_i \times \boldsymbol{v}_i$ is a vector perpendicular to the plane of $\boldsymbol{r}_i$ and $\boldsymbol{v}_i$; but, as the direction of the vector is constant in plane motion, the angular momentum in the simple case of plane motion may be treated as a scalar quantity.

[1] The vector product is used to render the product of sums of vectors amenable to the distributive laws and to avoid operating with sines.

We may substitute from (1) in (7) either for $\boldsymbol{r}_i$ or for $\boldsymbol{v}_i$ or for both. The substitution for $\boldsymbol{v}_i$ alone does not seem interesting. The substitution for $\boldsymbol{r}_i$ alone gives

$$\text{A. M.} = \sum_i m_i(\boldsymbol{s} + \boldsymbol{r}_i') \times \boldsymbol{v}_i = M\boldsymbol{s} \times \bar{\boldsymbol{v}} + \sum_i m_i \boldsymbol{r}_i' \times \boldsymbol{v}_i. \tag{8}$$

The interpretation is that the angular momentum about a point F may be computed as the sum of the angular momentum about F of the whole linear momentum $M\bar{\boldsymbol{v}}$ considered as located at O and of the angular momentum about O, but not relative to O (for the actual velocity $\boldsymbol{v}_i$ remains).

The substitution for both $\boldsymbol{r}$ and $\boldsymbol{v}$ gives

$$\begin{aligned}\text{A. M.} &= \sum_i m_i(\boldsymbol{s} + \boldsymbol{r}_i') \times (\boldsymbol{u} + \boldsymbol{v}_i') \\ &= M\boldsymbol{s} \times \boldsymbol{u} + \boldsymbol{s} \times (M\bar{\boldsymbol{v}}') + M\bar{\boldsymbol{r}}' \times \boldsymbol{u} + \sum_i m_i \boldsymbol{r}_i' \times \boldsymbol{v}_i'. \end{aligned} \tag{9}$$

Now $\boldsymbol{s} \times (M\boldsymbol{u})$ is the angular momentum about F of a mass M moving with O and $\Sigma m_i \boldsymbol{r}_i' \times \boldsymbol{v}_i'$ is the angular momentum about and relative to O. As $\bar{\boldsymbol{r}}'$ is the vector from O to G, $-\bar{\boldsymbol{r}}'$ is the vector from G to O, and $\bar{\boldsymbol{r}}' \times (M\boldsymbol{u})$ is the angular momentum, reversed in sign, of a mass M moving with O about G. Finally, $\boldsymbol{s} \times (M\bar{\boldsymbol{v}}')$ is the angular momentum about F of the relative linear momentum $M\bar{\boldsymbol{v}}'$ located at O. Hence

THEOREM 7. *The angular momentum of a system may be written as the sum of* FOUR *terms:* (*a*) *the angular momentum about F of the mass of the system located at any moving point O,* (*b*) *the relative angular momentum about O,* (*c*) *the reversed angular momentum of the mass of the system located at O about, but not relative to, the center of gravity,* (*d*) *the angular momentum about F of the relative momentum considered as located at O.*

If we take $O = G$, we have $\bar{\boldsymbol{r}}' = 0$ and $\bar{\boldsymbol{v}}' = 0$. Hence

THEOREM 8. *The angular momentum of a system about a fixed point F is the sum of the angular momentum of the mass of the system moving with the center of gravity and of the angular momentum about and relative to the center of gravity.*

Thus the use of the center of gravity serves to separate the translational and rotational parts of the motion for *both* angular momentum and kinetic energy.

To compute the angular momentum of a rigid lamina we *do not need* Theorem 7; we may apply the original definition (7) and its modification (8) which relates the angular momentum about two different points. For if the lamina has a motion of pure translation, we have A. M. $= M\bar{\boldsymbol{r}} \times \boldsymbol{v}$, where $\bar{\boldsymbol{r}}$ gives the position of the center of gravity and $\boldsymbol{v}$ is the velocity of any point. And if the lamina is rotating, the angular momentum about the instantaneous center is $I_C\omega\boldsymbol{n}$, where I_C is the moment of inertia about the center and $\boldsymbol{n}$ is a unit vector perpendicular to the plane; and the angular momentum about any point F is the sum of $I_C\omega\boldsymbol{n}$ and the angular momentum about F of the linear momentum $M\bar{\boldsymbol{v}}$ located at C.

In many cases, however, it is *convenient* to have an expression for the angular momentum about an arbitrary point F in terms of the motion of a point O fixed

in the lamina and of the motion relative to O. This is given by (9), in which some reductions may now be made. The term $\Sigma m_i \boldsymbol{r}_i' \times \boldsymbol{v}_i'$ becomes $I'\omega\boldsymbol{n}$. If θ be the angle from the direction of $\boldsymbol{s}$ to that of $\boldsymbol{u}$, the first term becomes $M\, s\, u \sin\theta\, \boldsymbol{n}$, where s and u are the magnitudes of the corresponding vectors. The relative velocity $\bar{\boldsymbol{v}}'$ has the magnitude $OG\omega$, and a direction perpendicular to OG. If φ be the angle from $\boldsymbol{s}$ to $OG = \bar{\boldsymbol{r}}'$, the angle from $\boldsymbol{s}$ to $\boldsymbol{v}'$ is $90° + \varphi$, and the term $M\boldsymbol{s} \times \bar{\boldsymbol{v}}'$ reduces to $MsOG\omega \cos\varphi\ \boldsymbol{n}$. If the angle from OG to u be ψ, the term $M\bar{\boldsymbol{r}}' \times \boldsymbol{u}$ is $M\, OG\, u \sin\psi\ \boldsymbol{n}$. Hence

$$\text{A. M.} = M(su \sin\theta + k'^2\omega + OG\omega s \cos\varphi + OG\, u \sin\psi), \qquad (9')$$

where $k'^2 = I'/M$ and where the unit vector $\boldsymbol{n}$ has been thrown out.

By introducing the instantaneous center C, we may write $u = CO\omega$. The angle ψ from OG to $\boldsymbol{u}$ is connected with the angle ψ' from CO to OG by the equation $\psi' + \psi = 90°$. Hence $\sin\psi = \cos\psi'$. Now $CO\cos\psi'$ and $s\cos\varphi = FO\cos\varphi$ are respectively the projections of CO and FO on OG. Hence

$$\text{A. M.} = [M(su \sin\theta + k'^2\omega)] + [M\, OG\, \omega(\text{proj. } FO + \text{proj. } CO)]. \qquad (9'')$$

The first bracket contains the terms which we should find for the angular momentum of a lamina if we followed the kinematic resolution of the motion into a translation of any point and a rotation about that point. The second bracket contains the correction terms necessary for the correct dynamic result.

As the sum of the projections is equal to the projection of the sum, we may form the vector $OF + OC = \boldsymbol{R}$, which runs from O to the middle point of FC and as far again. The correction terms then reduce to $-\, M\, OG\, \omega$ proj. $\boldsymbol{R}$, the projection of $\boldsymbol{R}$ being along OG. This will vanish when $\boldsymbol{R}$ is perpendicular to OG, that is, when O lies on a circle with G and the middle point of FC as diametrically opposite points. Hence

THEOREM 9. *The kinematic resolution of the motion gives the correct result for the angular momentum about a point F when and only when the chosen point O, fixed in the lamina, lies on a circle, passing through the center of gravity G and the middle point Q of the line joining the point F and the instantaneous center C, and having GQ as a diameter.*

THEOREM 10. *If a point of reference O is fixed in the lamina, there are ∞^1 points F about which the angular momentum reduces to the sum of two terms, $M(su \sin\theta + k'^2\omega)$, and the locus of these points is a straight line perpendicular to OG and so placed that the segments CF from the instantaneous center to F are bisected by the perpendicular to OG at O.*

This follows from the preceding theorem.

For motion in space the condition $\boldsymbol{s} \times \bar{\boldsymbol{v}}' + \bar{\boldsymbol{r}}' \times \boldsymbol{u} = 0$, which reduces (9) to two terms, is a vector equation. The equivalent three scalar equations have in general three simultaneous solutions, of which one gives the center of gravity and the other two (which may be real or imaginary) are complicated and apparently without interest. *The kinematic resolution is therefore of general dynamic value in space only when the point of reference is the center of gravity.*

Forces and Moments. If forces $\boldsymbol{F}_i$ act on the masses m_i, we have

$$m_i \frac{d\boldsymbol{v}_i}{dt} = \boldsymbol{F}_i \quad \text{and} \quad \sum_i m_i \frac{d\boldsymbol{v}_i}{dt} = M\frac{d\bar{\boldsymbol{v}}}{dt} = \sum_i \boldsymbol{F}_i. \tag{10}$$

This is the theorem on the motion of the center of gravity. Also

$$\sum_i m_i \boldsymbol{r}_i \times \frac{d\boldsymbol{v}_i}{dt} = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i,$$

Hence

$$\sum_i m_i \boldsymbol{s} \times \frac{d\boldsymbol{v}_i}{dt} + \sum_i m_i \boldsymbol{r}_i' \times \frac{d\boldsymbol{u}}{dt} + \sum_i m_i \boldsymbol{r}_i' \times \frac{d\boldsymbol{v}_i'}{dt} = \sum_i \boldsymbol{s} \times \boldsymbol{F}_i + \sum_i \boldsymbol{r}_i' \times \boldsymbol{F}_i$$

or

$$\boldsymbol{s} \times \left(M\frac{d\bar{\boldsymbol{v}}}{dt} \right) + M\bar{\boldsymbol{r}}' \times \frac{d\boldsymbol{u}}{dt} + \sum_i m_i \boldsymbol{r}_i' \times \frac{d\boldsymbol{v}_i'}{dt} = \boldsymbol{s} \times \sum_i \boldsymbol{F}_i + \sum_i \boldsymbol{r}_i' \times \boldsymbol{F}_i.$$

The first terms on the two sides cancel by (10). Hence

$$M\bar{\boldsymbol{r}}' \times \frac{d\boldsymbol{u}}{dt} + \sum_i m_i \boldsymbol{r}_i' \times \frac{d\boldsymbol{v}_i'}{dt} = \sum_i \boldsymbol{r}_i' \times \boldsymbol{F}_i. \tag{11}$$

But

$$\frac{d}{dt}(\boldsymbol{r}_i' \times \boldsymbol{v}_i') = \frac{d\boldsymbol{r}_i'}{dt} \times \boldsymbol{v}_i' + \boldsymbol{r}_i' \times \frac{d\boldsymbol{v}_i'}{dt} = \boldsymbol{r}_i' \times \frac{d\boldsymbol{v}_i'}{dt}.$$

Hence

$$M\bar{\boldsymbol{r}}' \times \frac{d\boldsymbol{u}}{dt} + \frac{d}{dt}\sum_i m_i \boldsymbol{r}_i' \times \boldsymbol{v}_i' = \sum_i \boldsymbol{r}_i' \times \boldsymbol{F}_i. \tag{12}$$

THEOREM 11. *The moment of the forces about any point O is equal to the rate of change of angular momentum about and relative to O plus the mass of the system times the vector product of the relative coordinate* $\bar{\boldsymbol{r}}' = OG$, *of the center of gravity, and the acceleration of O.*

This theorem is equally applicable to the plane and to space, to a rigid and to a non-rigid body. If there are internal actions and reactions between the particles of the system in the lines joining the particles, these actions and reactions drop out. The theorem includes Huntington's theorems. We may state as corollaries:

THEOREM 12. *The moment of the (external) forces applied to the system is equal to the rate of change of angular momentum when and only when the point of reference (a) is the center of gravity, (b) is fixed in space, (c) is moving with a uniform velocity, or (d) has a vector acceleration passing through the center of gravity.*

THEOREM 13. *If the system is a rigid lamina and the point of reference is fixed in the body and satisfies the conditions of Theorem* 12, *then*

$$I'\frac{d\omega}{dt} = \text{moment of forces.} \tag{13}$$

In the case of a rigid body in space the angular momentum is a linear vector function (relative to axes fixed in the body) of the angular velocity regarded as a vector, and this must be considered in forming the equation corresponding to (13).

THE THEOREM OF MOMENTS.

By DR. C. G. KNOTT, University of Edinburgh.

Professor Huntington's recent communication on the "Theorem of Moments" (in the AMERICAN MATHEMATICAL MONTHLY for December, 1914) calls attention to a point of real importance. He designedly confined his discussion to the elementary case of uniplanar rigid dynamics; but it struck me that the extraordinarily simple proof of the general case in terms of quaternion analysis might be of interest to readers of the MONTHLY.

I assume Newton's second and third laws of motion or their equivalents.

Let the mass m in position ρ be acted on by the force β. Then for one particle[1]

$$m\ddot{\rho} = \beta$$

and for a system of particles

$$\Sigma m\ddot{\rho} = \Sigma\beta \tag{1}$$

in which $\Sigma\beta$ includes only the applied or external forces (Lex III).

Also, introducing each corresponding ρ as multiplier, and summing for the system, we have

$$\Sigma m\rho\ddot{\rho} = \Sigma\rho\beta. \tag{2}$$

This breaks up into the scalar and vector parts

$$\Sigma Sm\rho\ddot{\rho} = \Sigma S\rho\beta$$

and

$$\Sigma Vm\rho\ddot{\rho} = \Sigma V\rho\beta. \tag{3}$$

The first of these (2) is the "virial" equation and was first used by Hamilton.

The latter equation (3) expresses the theorem of moments; and in it also, by virtue of Lex III, $\Sigma V\rho\beta$ includes only the applied forces.

Now let π be the vector position of any point at rest or in motion, and let σ be the vector position with respect to this new point of any element of the system. Then

$$\rho = \pi + \sigma, \qquad \ddot{\rho} = \ddot{\pi} + \ddot{\sigma}.$$

Equations (1) and (2) become

$$\Sigma m(\ddot{\pi} + \ddot{\sigma}) = \Sigma\beta, \tag{1'}$$

$$\Sigma m(\pi + \sigma)(\ddot{\pi} + \ddot{\sigma}) = \Sigma(\pi + \sigma)\beta. \tag{2'}$$

Since π is the vector position of a definite point, it may be taken outside or placed within the summation symbol at will. Multiplying π into (1') and subtracting from (2') we find

$$\Sigma m\sigma(\ddot{\pi} + \ddot{\sigma}) = \Sigma\sigma\beta$$

[1] See my edition (1904) of Kelland & Tait's *Introduction to Quaternions*, page 160.

and the vector part of this is

$$V(\Sigma m\sigma)\ddot{\pi} + \Sigma V m\sigma\ddot{\sigma} = \Sigma V\sigma\beta. \tag{4}$$

Now if the point π is to be a point with regard to which we may apply the theorem of moments in the form (3), we must have

$$\Sigma V m\sigma\ddot{\sigma} = \Sigma V\sigma\beta \tag{5}$$

and hence the condition to be satisfied by π is simply

$$V(\Sigma m\sigma)\ddot{\pi} = 0. \tag{6}$$

But, in general, $\Sigma m\sigma = \mu\Sigma m$, where μ is the vector position of the center of mass of the system referred to the point π. Hence (6) becomes

$$V\mu\ddot{\pi} = 0. \tag{7}$$

This equation may be satisfied by any one of the conditions

$$(1)\ \mu = 0; \qquad (2)\ \ddot{\pi} = 0; \qquad (3)\ \ddot{\pi} \text{ parallel to } \mu.$$

Hence we may legitimately use the point whose vector is π as a pivot for moments when

(1) it is the center of mass; or
(2) it has no acceleration so that its velocity is constant (the case of) rest or uniform motion in a straight line; or
(3) its acceleration is directed towards the center of mass.

Since the acceleration of the center of mass is necessarily directed through its own position, condition (3) really includes (1). And thus, quite generally, the theorem of moments may be applied with reference to any point whose acceleration is either zero or directed through the center of mass.

BOOK REVIEWS

MATHEMATICS IN DR. ELIOT'S FIVE-FOOT SHELF OF BOOKS.

By W. H. BUSSEY, University of Minnesota.

Several years ago the attention of the whole country was attracted by the announcement made by Dr. Charles W. Eliot that he proposed to select a group of books, filling a five-foot shelf, in the belief that "the faithful and considerate reading of these books, with such re-readings and memorizings as individual taste may prescribe, will give any man the essentials of a liberal education, even if he can devote to them but fifteen minutes a day." The selection was made by him in due time, and the books have since been published by P. F. Collier and Son in a fifty volume set called "The Harvard Classics." In the introduction

to the set, Dr. Eliot states his purpose as follows: "My purpose in selecting The Harvard Classics was to provide the literary materials from which a careful and persistent reader might gain a fair view of the progress of man, observing, recording, inventing, and imagining, from the earliest historical times to the close of the nineteenth century."

The purpose of this review is not to discuss the merits of this five-foot shelf of books, but rather to call to the attention of the readers of the MONTHLY those parts of the set which are of special interest to mathematicians. The set is supposed to furnish a careful reader with the essentials of a liberal education, and surely no such thing is possible without some acquaintance with mathematics and the related sciences, physics and astronomy. Dr. Eliot says in the introduction that he "proposed to make such a selection as any intellectually ambitious American family might use to advantage, even if their early opportunities of education had been scanty."

In such a set of books one would not expect to find any technical mathematical works like Euclid's "Elements" or Newton's "Principia," but one would expect to find something about the great realm of thought which has been explored by means of mathematics.

Volume 30, entitled "Scientific Papers," contains an interesting essay by Simon Newcomb on "The Extent of the Universe." It is preceded by a short introductory note telling something of Newcomb's life and work. The essay itself and the following quotation from the introductory note give some idea of how the mathematical sciences are represented in The Harvard Classics.

"Newcomb's chief labors were in the department of mathematical astronomy, and were directed toward the explanation of the observed movements of the heavenly bodies. The difficulty and complexity of the calculations involved are beyond the conception of the layman; and the achievements which brought Newcomb honors from the learned of almost all civilized countries have to be taken on trust by the general public. He had, nevertheless, an admirable power of clear exposition of those parts of his subject which were capable of popularization; and the accompanying paper is a good example of the simple treatment of a large subject."

More information about the editor's intention concerning scientific work is gained from the following quotations from the introduction.

"Nothing has been included in the series which does not possess good literary form." "It was hard to make up an adequate representation of the scientific thought of the nineteenth century, because much of the most productive scientific thought has not yet been given a literary form. The discoverers' original papers on chemistry, physics, geology, and biology have usually been presented to some scientific society, and have naturally been expressed in technical language, or have been filled with details indispensable from the scientific point of view but not instructive for the public in general."

There is a general index of more than 300 pages in volume 50, and the writer of this review naturally looked to see what he could find under the index word *mathematics*. The result was disappointing, not because the word was missing but because many of the references were trivial. For example, the reference to "Franklin on mathematicians" was to the following paragraph of his "Autobiography" which describes one of the members of a club organized by Franklin.

"Thomas Godfrey, a self-taught mathematician, great in his way, and afterward inventor of what is now called Hadley's Quadrant. But he knew little out of his way, and was not a pleasing

companion; as, like most great mathematicians I have met with, he expected universal precision in everything said, or was forever denying or distinguishing upon trifles, to the disturbance of all conversation."

Having found several references just as trivial, the writer decided that either the set of books did not contain much of special interest to mathematicians or the maker of the index had failed to find such material. The writer then looked through the set carefully for that which he had in mind to find, and discovered the following things of interest.

Volume 30 has already been referred to as containing an essay by Simon Newcomb, who was president of the American Mathematical Society for the two years 1897–98. Every American mathematician knows something of his work and ought to be interested in this paper on "The Extent of the Universe" (Vol. 30, pages 323–336). It contains no mathematics except some ordinary arithmetic and a fact or two from solid geometry. In this same volume, which is entitled "Scientific Papers," are the following papers of interest to mathematicians: Six lectures by Faraday on "The Forces of Matter" (Vol. 30, pages 1–88), especially lectures 1 and 2 on gravitation; a lecture by Helmholtz on "The Conservation of Force" (Vol. 30, pages 180–220); and two lectures by Lord Kelvin on "The Wave Theory of Light" (Vol. 30, pages 263–286), and "The Tides" (Vol. 30, pages 287–321).

Volume 34, entitled "French and English Philosophers," contains a number of interesting things. First of all there is DesCartes's well-known "Discourse on Method" (Vol. 34, pages 5–62) which is preceded by a short biography of DesCartes. Mathematicians will be especially interested in Part 1 in which are found various considerations touching the sciences.

Next come Voltaire's "Lettres Philosophiques" which are also known as "Letters on the English" (Vol. 34, pages 65–162). They represent Voltaire's scientific and philosophic interests. Letter 12 is a letter on Lord Bacon. It begins with the following eulogy of Sir Isaac Newton.

"Not long since the trite and frivolous question following was debated in a very polite and learned company, viz., who was the greatest man, Cæsar, Alexander, Tamerlane, Cromwell, etc.? Somebody answered that Sir Isaac Newton excelled them all. The gentleman's assertion was very just; for if true greatness consists in having received from heaven a mighty genius, and in having employed it to enlighten our own mind and that of others, a man like Sir Isaac Newton, whose equal is hardly found in a thousand years, is a truly great man. And those politicians and conquerors (and all ages produce some) were generally so many illustrious wicked men. That man claims our respect who commands over the minds of the rest of the world by the force of truth, not those who enslave their fellow creatures: he who is acquainted with the universe, not they who deface it. Since, therefore, you desire me to give you an account of the famous personages whom England has given birth to, I shall begin with Lord Bacon, Mr. Locke, Sir Isaac Newton, etc. Afterwards the warriors and ministers of state shall come in their order."

The next letter (No. 13) "On Mr. Locke" contains this comment on DesCartes: "Our DesCartes, born to discover the errors of antiquity, and at the same time to substitute his own." Voltaire is here referring to DesCartes' theory of vortices about which he has more to say in the next two letters, No. 14 "On DesCartes and Sir Isaac Newton" and No. 15 "On Attraction." The first of

these two contrasts the lives and rival theories of DesCartes and Newton. It contains considerable biography of both men. Voltaire refers to Newton as follows: "This famous Newton, this destroyer of the Cartesian system, died in March, 1727. His countrymen honoured him in his life time, and interred him as though he had been a king who had made his people happy." The latter half of the letter is a defense of DesCartes against those who, in their enthusiasm for Newton, were inclined to belittle him and all his work. Voltaire has this to say in reply to one who presumed to say that DesCartes was not a great geometrician:

"Those who make such a declaration may justly be reproached with flying in their master's face. DesCartes extended the limits of geometry as far beyond the place where he found them as Sir Isaac did after him. The former first taught the method of expressing curves by equations. This geometry which, thanks to him for it, is now grown common, was so abstruse in his time, that not so much as one professor would undertake to explain it; and Schotten in Holland and Fermat in France were the only men who understood it."

The letter "On Attraction" begins as follows:

"The discoveries which gained Sir Isaac Newton so universal a reputation relate to the system of the world, to light, to geometrical infinities, and lastly to chronology, with which he used to amuse himself after the fatigue of his severer studies. I will now acquaint you (without prolixity f possible) with the few things which I have been able to comprehend of all these sublime ideas."

This paragraph is really an introduction to three letters of which the one "On Attraction" is the first. Letter 16 is a popular account of "Sir Isaac Newton's Optics" and Letter 17 is "On Infinities in Geometry and Sir Isaac Newton's Chronology." The phrase "Infinities in Geometry" refers to differential and integral calculus. In the letter "On Attraction" Voltaire tells the well known story of the falling apple starting in Newton's mind the train of thought which led him to his theory of gravitation.

The following quotation from the beginning of Letter 17 tells what kind of impression the infinitesimal calculus made on a layman:

"The labyrinth and abyss of infinity is also a new course Sir Isaac Newton has gone through, and we are obliged to him for the clue by whose assistance we are enabled to trace its various windings. DesCartes got the start of him also in this astonishing invention. He advanced with mighty steps in his geometry, and was arrived at the very borders of infinity, but went no farther. Dr. Wallis, about the middle of the last century, was the first to reduce a fraction by perpetual division to an infinite series. The Lord Brouncker employed this series to square the hyperbola. Mercator published a demonstration of this quadrature; about which time Sir Isaac Newton, being then twenty-three years of age, had invented a general method to perform on all geometrical curves what had just before been tried on the hyperbola. It is to this method of subjecting everywhere infinity to algebraical calculations that the name is given of differential calculations or of fluxions and integral calculation. It is the art of numbering and measuring exactly a thing whose existence cannot be conceived."

Voltaire has this to say about the controversy over the invention of the calculus:

"For many years the invention of this famous calculation was denied to Sir Isaac Newton. In Germany Mr. Leibnitz was considered as the inventor of the differences or moments, called fluxions, and Mr. Bernoulli claimed the integral calculus. However, Sir Isaac is now thought to have first made the discovery, and the other two have the glory of having once made the world doubt whether it was to be ascribed to him or to them."

The six letters of Voltaire's which have been mentioned make 35 pages of very interesting reading. While reading them, one should bear in mind the dates of the three men, DesCartes (1596–1650), Newton (1642–1727), and Voltaire (1694–1778).

Volume 39, entitled "Famous Prefaces," contains the "Dedication of the Revolutions of the Heavenly Bodies" by Nicholaus Copernicus (1543). The book which was introduced by this dedication laid the foundation of modern astronomy. At the time when it was written, the earth was believed by all to be the fixed center of the universe; and although many of the arguments used by Copernicus were invalid and absurd, he was the first modern to put forth the heliocentric theory as "a better explanation." It remained for Kepler, Galileo and Newton to establish the theory on firm grounds. The dedication is to Pope Paul III. Copernicus explains how he came to write the book and the long delay in publishing it. This volume of The Harvard Classics also contains the preface to Newton's "Philosophiæ Naturalis Principia Mathematica." Mathematicians of to-day do not need to read this work of Newton's as a part of their mathematical education because they can get the substance of it from later works, but they will find it worth while to read what Newton had to say about it in his preface.

Volume 48 is devoted entirely to the works of Pascal, but of course it is Pascal the philosopher rather than Pascal the mathematician. It contains his "Thoughts" and some of his letters and minor works. Section 1 of the "Thoughts" is entitled "Thoughts on mind and style." Some of these refer to mathematics. In fact Thought No. 1 is on "The difference between the mathematical and the intuitive mind." Section 6 is on "The Philosophers." Here is a quotation from this section, being Thought No. 340. "The arithmetical machine produces effects which approach nearer to thought than all the actions of animals. But it does nothing which would enable us to attribute will to it, as to the animals." Later in the volume there is a letter from Pascal to Queen Christina of Sweden on sending her the arithmetical machine. The following quotation is from Thought No. 908 of Section 14 entitled "Polemical Fragments." "But is it probable that probability gives assurance?" Of course Pascal was not thinking of the mathematical theory of probability when he wrote this epigram, for that branch of mathematics was only in its infancy in his day; but we may well ask the question with reference to some of the conclusions which have been drawn from the theory of probability.

Of Pascal's minor works included in this volume, the only one of special interest to mathematicians is an essay "On the Geometrical Spirit." It consists of an introduction on page 427; Section I which is "On the method of geometrical, that is, of methodical and perfect demonstrations" on pages 428–444; and Section II on "The Art of Persuasion," which the editor has, for some reason, printed as a separate work on pages 406–417. In addition to what the title would lead one to expect, this work "On the Geometrical Spirit" contains a long argument against those who believed in "indivisibles." At the end of his argument Pascal says that "those who will not be satisfied with these reasons, and will

persist in the belief that space is not divisible *ad infinitum*, can make no pretensions to geometrical demonstrations, and although they may be enlightened in other things, they will be very little in this; for one can easily be a very capable man and a bad geometrician." In the course of his argument, Pascal has some interesting things to say about the relations of unity and zero to the number system. Near the beginning of Section I, Pascal has considerable to say about definitions and the necessity of leaving some things in geometry undefined. He has this to say of those who insist on defining everything:

"For there is nothing more feeble than the discourse of those who wish to define these primitive words (space, time, motion, number, equality). What necessity is there, for example, of explaining what is understood by the word *man*? Do we not know well enough what the thing is that we wish to designate by the term? And what advantage did Plato think to procure us by saying that he was a two-legged animal without feathers? As though the idea that I have of him naturally, and which I cannot express, were not clearer and surer than that which he gives me by his useless and even ridiculous explanation; since a man does not lose humanity by losing the two legs, nor does a capon acquire it by losing his feathers."

Volume 48 closes with Pascal's "Preface to the Treatise on Vacuum" and a very short "New Fragment of the Treatise on Vacuum." These have reference to Pascal's experimental work on atmospheric pressure. He repeated the experiments of Torricelli by which the pressure of the atmosphere could be estimated as a weight; and by what has been called "The Great Experiment of the Puy-de-Dôme" he confirmed Torricelli's theory of the cause of barometrical variations. In this experiment Pascal obtained at the same instant readings of the height of a column of mercury in a barometer tube at different altitudes.

The parts of The Harvard Classics which have been mentioned as representing the mathematical sciences are all contained in the four volumes 30, 34 39, 48. The number of pages in these parts is more than 250. The authors represented are Newcomb, Faraday, Kelvin, Helmholtz, DesCartes, Voltaire, Copernicus, Newton, and Pascal. The parts written by Voltaire tell at some length of the work of DesCartes and Newton, and contain references to Bernoulli, Leibnitz, Wallis, Fermat and Mercator. Thus it is seen that the mathematical sciences have not been altogether neglected. An interesting question for readers of the MONTHLY is this: What else relating to the mathematical sciences might Dr. Eliot have included in his "Five-Foot Shelf of Books"?

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

PROBLEMS FOR SOLUTION.

ALGEBRA.

When this issue was made up, no solutions had been received for number 406.

436. Proposed by WALTER H. DRANE, Cumberland University, Tenn.

The product of two numbers p and q may be obtained by dividing p by 2 successively, discarding remainders, until the quotient 1 is obtained and then multiplying q by 2 successively and

adding those products, $2^i q$, which correspond to those quotients of p that are odd numbers.

E. g. 51×81. 25, 162; 12, 324; 6, 648; 3, 1,296; 1, 2,592.

$$51 \times 81 = 81 + 162 + 1{,}296 + 2{,}592 = 4{,}131.$$

Prove or disprove the truth of the above proposition.

437. Proposed by C. N. SCHMALL, New York City.

Given that $S_1, S_2, S_3, \cdots, S_k$ are the sums of k arithmetical series, each taken to n terms. The first terms are respectively 1, 2, 3, $\cdots$, k, and the common differences are 1, 3, 5, $\cdots$, $(2k-1)$. Show that

$$S_1 + S_2 + S_3 + \cdots + S_k = \frac{nk(nk+1)}{2}.$$

GEOMETRY.

When this issue was made up, no solutions had been received for numbers 446 and 449.

466. Proposed by HORACE OLSON, Chicago, Illinois.

Given the edges of a triangular pyramid, find the radius of the inscribed sphere.

467. Proposed by E. T. BELL, Seattle, Washington.

It is well-known that if i, j, k, l are concyclic points, W_i the Wallace line (frequently, and erroneously, called the Simson line), of i with respect to the triangle jkl, then W_i, W_j, W_k, W_l are concurrent, say in the point $\{i, j, k, l\}$. If 1, 2, 3, $\cdots$ denote concyclic points, prove that:

(i) $\{1, 2, 3, 4\}$, $\{1, 2, 3, 5\}$, $\{1, 2, 4, 5\}$, $\{1, 3, 4, 5\}$, $\{2, 3, 4, 5\}$ are concyclic; say on the circle [1, 2, 3, 4, 5];

(ii) Starting with 1, 2, 3, 4, 5, 6, omitting each point in turn, by (i), six circles, are found; these are concurrent, say in the point $\{1, 2, 3, 4, 5, 6\}$;

(iii) Starting with 1, 2, 3, 4, 5, 6, 7, seven points of the kind in (ii) are found; these lie on a circle.

(iv) Continuing thus indefinitely, there is, in each case, finally a unique point or circle according as the number of initial points is even or odd. Also, at any stage, the point of concurrence or the circle bears a simple relation to the initial points: what is it?

[*Note.*—This problem is obviously connected with the theorems of Clifford (*Mathematical Papers*, pp. 38 and 410), but it is interesting to note that it may be completely solved by the methods of Euclid, Books I to III (but at considerable length), and hence is well within the range of high school students. But a very simple proof may be given by the methods of strictly elementary analytical geometry.]

468. Proposed by ELMER SCHUYLER, Brooklyn, New York.

Given two circles and a straight line, to draw a circle tangent to the line and coaxial with the two given circles.

469. Proposed by J. ALEXANDER CLARKE, West Philadelphia High School.

If in an isosceles triangle, a circle is described on one side as diameter, and a line is drawn through the mid-point of the side parallel to the base, the circle and the parallel will intercept on the *tri*sector of the angle at the vertex a segment equal to the radius of the circle. Show how this can be used to trisect any angle.

CALCULUS.

When this issue was made up, no solutions had been received for numbers 336, 338–340, 342, 348, 353, 360, and 363.

387. Proposed by C. N. SCHMALL, New York City.

Show that the volume bounded by the cone $x^2 + y^2 = (a - z)^2$ and the planes $x = 0$, $x = z$, is $\frac{2}{3}a^3$.

388. Proposed by PAUL CAPRON, U. S. Naval Academy.

If $f(x, y) = 0$ represents (in rectangular coordinates) a curve having a simple tangency to the axis of x at the origin, the value of $x^2/2y$, derived from $f(x, y) = 0$, and evaluated for $x = 0$, $y = 0$, will be the radius of curvature at the origin; or if the curve is similarly tangent to the y-axis at the origin, $y^2/2x$, evaluated for $x = 0$, $y = 0$, is the radius of curvature at the origin.

389. Proposed by FRANK R. MORRIS, Glendale, Calif.

A man is at the southeast corner of a section of land and wishes to walk to the opposite corner in the least possible time. A circular track with a radius of $1/\pi$ miles is located in the section tangent to the west line at a point 120 rods from the south line. Conditions are such that he can walk at the rate of 4 miles an hour inside the track and 3 miles an hour outside the track. What course should he choose and how long is it?

MECHANICS.

When this issue was made up, no solutions had been received for numbers 274, 277, 279, 287, 291, and 292.

311. Proposed by B. J. BROWN, Student in Drury College.

A particle oscillates in a straight line about a center of force varying as the distance, and is subject to a retardation $k \times (\text{vel.})^2$. If a, b be two successive elongations, on opposite sides, prove that $(1 + 2ka)e^{-2ka} = (1 - 2kb)e^{2kb}$. What form does the result take if a is infinite? From *Lamb's Dynamics*, p. 299, ex. 14.

312. Proposed by C. N. SCHMALL, New York City.

A ball of elasticity e is projected upward from a point on an inclined plane, so that after its first contact with the plane it rebounds to its starting point. If ϕ be the inclination of the plane to the horizontal, and ψ the angle made by the line of projection with the inclined plane, show that

$$\cot \phi \cot \psi = e + 1.$$

313. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

A heavy extensible wire of length c and of constant cross-section w, and density k, is suspended by one end and hangs vertically. If e is the coefficient of elasticity, show that the length of the wire when stretched will be $c(1 + ekgw/2)$.

NUMBER THEORY.

When this issue was made up, no solutions had been received for numbers 188–9, 191–2, 196, 200, 205, 208–9, 211, 214–15, 217, and 219.

232. Proposed by E. T. BELL, Seattle, Washington.

If $F(x)$ is any function of x which vanishes with x, and which, for $0 < |x| \leqq |\xi|$, can be expanded in an absolutely convergent series of positive powers of x, show that a function $f(n)$ may be found, essentially in one way only, such that

$$\int_0^{\xi} \frac{1}{x} F(x)dx = -\log \prod_{n=1}^{\infty} (1 - \xi^n)^{(1/n)f(n)},$$

and find the form of $f(n)$ explicitly in terms of the coefficients in the expansion of $F(x)$. Hence, as particular examples, expand (when possible by this method) e^x as an infinite product, and show that

$$\frac{1}{e} = \prod_{n=1}^{\infty} \left(1 - \frac{1}{2^n}\right)^{(1/n)\phi(n)}$$

where $\varphi(n)$ is the totient of n.

233. Proposed by V. M. SPUNAR, Chicago, Illinois.

Solve in rational numbers $x^2 + y^2 = a^2$, $xy = m/n$, where m and n are integers and relatively prime to each other.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

424. Proposed by S. A. JOFFE, New York City.

Sum the series

$$\binom{n}{a} - \binom{n-1}{a}\binom{i}{1} + \binom{n-2}{a}\binom{i}{2} - \binom{n-3}{a}\binom{i}{3} + \cdots + (-1)^i \binom{n-i}{a}$$

and consider the cases $i = a$ and $i > a$.

I. Solution by Frank Irwin, University of California.

The sum is $\binom{n-i}{a-i}$. This will hold for all (positive) values of n, a, i, provided we put $\binom{k}{l} = 0$ if $k < l$ or if l is negative. Thus for $i > a$, the sum will be zero; while for $n - i < a$, some of the terms of the sum will drop out.

First Proof.—These statements may be proved by mathematical induction. For they are true when $i = 1$; suppose them then to be true for some particular value of i and all values of n. We shall show that they are true for $i + 1$. For we have:

$$\binom{n}{a} - \binom{n-1}{a}\binom{i}{1} + \cdots \pm \binom{n-k}{a}\binom{i}{k} + \cdots = \binom{n-i}{a-i},$$

$$\binom{n-1}{a} - \binom{n-2}{a}\binom{i}{1} + \cdots \mp \binom{n-k}{a}\binom{i}{k-1} + \cdots = \binom{n-i-1}{a-i};$$

whence, subtracting, we obtain

$$\binom{n}{a} - \binom{n-1}{a}\binom{i+1}{1} + \cdots \pm \binom{n-k}{a}\left[\binom{i}{k} + \binom{i}{k-1}\right] + \cdots = \binom{n-i}{a-i} - \binom{n-i-1}{a-i},$$

or

$$\binom{n}{a} - \binom{n-1}{a}\binom{i+1}{1} + \cdots \pm \binom{n-k}{a}\binom{i+1}{k} + \cdots = \binom{n-i-1}{a-i-1},$$

the desired result.

Second Proof.—$\binom{n-i}{a-i}$ gives the combinations of $n - i$ objects, say white balls, $a - i$ at a time. We may count these combinations in another way. Let us add to the white balls i black balls, and consider the $\binom{n}{a}$ combinations of black and white balls together, a at a time. If we take one of these combinations *that contains all the black balls* and discard the latter, we get one of the combinations of the white balls $a - i$ at a time; and each of these latter combinations may be obtained in this way. To count these latter, then, we must subtract from the $\binom{n}{a}$ combinations above those that omit any of the black balls. There are $\binom{n-1}{a}$ that omit a particular black ball, $\binom{i}{1}$ black balls; but the number

$$\binom{n}{a} - \binom{n-1}{a}\binom{i}{1}$$

is too small, since a combination that omits more than one black ball has been subtracted, in the second term, more than once. It is easily seen how, proceeding along these lines, we obtain our given sum

$$\binom{n}{a}-\binom{n-1}{a}\binom{i}{1}+\binom{n-2}{a}\binom{i}{2}-\cdots,$$

and it remains to examine how often herein a particular combination, say one that omits k black balls, has been reckoned. A little consideration will show that it has been counted $1-\binom{k}{1}+\binom{k}{2}-\cdots\pm 1=0$ times; so that the only combinations counted are those that contain all i black balls, and our sum is equal to the number of these combinations, that is, to $\binom{n-i}{a-i}$, as was to be proved.

When $i>a$, $a-i$ is negative and therefore $\binom{n-i}{a-i}=0$.

II. Solution by the Proposer.

By an elementary theorem in the calculus of finite differences, we have

$$\Delta^i u_x = u_{x+1}-\binom{i}{1}u_{x+i-1}+\binom{i}{2}u_{x+i-2}-\cdots+(-1)^i u_x.$$

Making here $x+i=n$, or $x=n-i$, we obtain

$$u_n-\binom{i}{1}u_{n-1}+\binom{i}{2}u_{n-2}-\cdots+(-1)^i u_{n-i}=\Delta^i u_{n-i}.$$

If we take

$$u_n=\binom{n}{a},\ \text{so that}\ u_{n-1}=\binom{n-1}{a},\ u_{n-2}=\binom{n-2}{a},\ \text{etc.},$$

the last equation becomes

$$\binom{n}{a}-\binom{i}{1}\binom{n-1}{a}+\binom{i}{2}\binom{n-2}{a}-\cdots+(-1)^i\binom{n-i}{a}=\Delta^i\binom{n-i}{a},$$

the first member of which is the given series.

In order to evaluate the second member, we notice that

$$\Delta\binom{x}{a}=\binom{x+1}{a}-\binom{x}{a}.$$

But as

$$\binom{x+1}{a}-\binom{x}{a}=\frac{(x+1)!}{(x+1-a)!a!}-\frac{x!}{(x-a)!a!}=\frac{x!}{(x+1-a)!(a-1)!}=\binom{x}{a-1},$$

therefore

$$\Delta\binom{x}{a}=\binom{x}{a-1},\quad \Delta^2\binom{x}{a}=\Delta\binom{x}{a-1}=\binom{x}{a-2},\quad \Delta^3\binom{x}{a}=\binom{x}{a-3},\quad \text{etc.}$$

and in general

$$\Delta^i\binom{x}{a}=\binom{x}{a-i};\quad \Delta^i\binom{n-i}{a}=\binom{n-i}{a-i}.$$

Hence

$$\binom{n}{a} - \binom{i}{1}\binom{n-1}{a} + \binom{i}{2}\binom{n-2}{a} - \cdots + (-1)^i \binom{n-i}{a} = \binom{n-i}{a-i}. \quad (1)$$

In the case when $i = a$, we have $a - i = 0$, and $\binom{n-i}{a-i} = 1$; therefore, equation (1) becomes

$$\binom{n}{a} - \binom{a}{1}\binom{n-1}{a} + \binom{a}{2}\binom{n-2}{a} - \cdots + (-1)^a \binom{n-a}{a} = 1. \quad (2)$$

When $i > a$, then $a - i$ is negative, so that $\binom{n-i}{a-i} = 0$. Hence for $i > a$,

$$\binom{n}{a} - \binom{i}{1}\binom{n-1}{a} + \binom{i}{2}\binom{n-2}{a} - \cdots + (-1)^i \binom{n-i}{a} = 0. \quad (3)$$

An excellent solution by a somewhat different method from the ones exhibited in these two solutions was received from A. M. HARDING.

425. Proposed by CLIFFORD N. MILLS, Brookings, S. D.

Solve for x and y the equations $2^{x+y} = 6$, $2^{x+1} = 3^y$.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

We shall agree to mean by a^x, where a is real, $e^{x \log a}$, in which the real logarithm of a is to be taken, so that a^x is single valued. We have, then, from the given equations, $(x + y)\log 2 = \log 6$, $(x + 1)\log 2 = y \log 3$, where all the logarithms are real. Solving these linear equations for x and y, we obtain the required solutions

$$x = \frac{(\log 6)^2 - \log 2 \cdot \log 12}{\log 2 \log 6} = 1.198 \text{ and } y = \frac{\log 12}{\log 2} = 1.387.$$

On account of the homogeneity of these fractions in the logarithmic function, we may substitute logarithms to the base 10 in our computation of the numerical values of x and y.

Also solved by EMMA GIBSON, NATHAN ALTSHILLER, A. L. MCCARTY, A. M. HARDING, HORACE OLSON, RICHARD MORRIS, G. W. HARTWELL, C. E. GITHENS, FRANK IRVIN, ELIZABETH B. DAVIS, F. L. CARMICHAEL, EDWARD S. INGHAM, S. A. JOFFE, W. C. EELLS, ELMER SCHUYLER, V. M. SPUNAR, R. M. MATHEWS, and CYRIL A. NELSON.

GEOMETRY.

454. Proposed by LOUIS ROUILLION, Mechanics Institute, New York City.

Show how to construct an equilateral triangle with its vertices lying on three [parallel] lines not equally spaced.

I. SOLUTION BY C. N. SCHMALL, New York City.

The proposer evidently meant that the three lines are parallel.

(i) Let x, y, and z be the three parallel lines in order. At any point, M, on the line y, construct the angles AMC and BMC each equal to 60° (Fig. 1). Draw the line AB, and, about the triangle AMB, circumscribe the circle ABC.

Draw the lines AC and BC. Then, the triangle ABC is equilateral as is easily seen and, therefore, satisfies the condition of the problem.

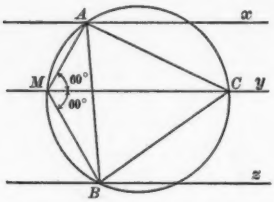

Fig. 1.

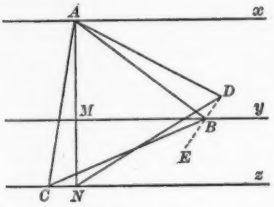

Fig. 2.

(ii) We may also construct the triangle as follows: Through the point, M (Fig. 2), draw AN perpendicular to the line y, and on AN as a side construct the equilateral triangle AND. Through D, draw ED perpendicular to AD, meeting the line y in B. Draw the line AB. Then draw AC, making the angle NAC equal to the angle BAD. Draw BC. Then the triangle ABC is equilateral, and has its vertices on the three given lines as is required. For, the two right triangles BAD and CAN are congruent, having, by construction, a side and an angle of one made equal to a side and an angle of the other. Hence, side AC equals side AB. Angle CAB equals angle $NAD = 60°$. Hence, angles ACB and ABC are equal and the triangle is equilateral.

II. Solution by Nathan Altshiller, University of Washington.

Let us generalize the problem in the following way:

Construct a triangle ABC so that its vertices A, B, C shall lie respectively on three given lines p, q, r, while its angles α, β, γ shall have each a given value.

The condition that the triangle shall have three given angles is equivalent to the condition that one of its angles, say α, shall have a given value, and that the ratio AB/AC, of the sides AB and AC including this angle, shall be equal to a given quantity k.

Let D be the foot of the perpendicular AD dropped from A upon the line r. Rotate the triangle ACD bodily about the point A into the position $AC'D'$ such that the line AC shall take the direction of the line AB. The angle DAD' is equal to the angle $CAB = \alpha$.

The perpendicular BE dropped from B upon AD', determines on AD' a point E such that

$$\frac{AE}{AD'} = \frac{AB}{AC'} = \frac{AB}{AC} = k.$$

This leads to the following method of solving the problem:

From any point A on p drop the perpendicular AD upon r and construct the line AD' making with AD an angle equal to the given angle α. On AD' take a

segment $AD' = AD$ and determine on it the point E so that $\frac{AE}{AD'} = k$. The perpendicular EB to AD' at the point E will meet q in a point B which is the second vertex, besides A, of the required triangle. The third vertex C is easily found.

This construction is applicable whether the given lines are concurrent, parallel, or form a triangle.

To satisfy the conditions of the generalized problem, it is necessary to exercise care in effecting the rotation in the proper sense. If the problem should read that ABC is to be similar to a given triangle without specifying which angle shall have its vertex on which of the given lines, the point A would be the origin of three triangles each solving the problem, but differently situated. The three triangles coincide, if the angles of ABC are to be equal to each other, as in the case of the proposed problem.

The particular case was also solved by DANIEL KRETH, ELIZABETH B. DAVIS, CLIFFORD N. MILLS, S. A. JOFFE, ELMER SCHUYLER, J. W. CLAWSON, and the PROPOSER.

455. Proposed by R. P. BAKER, University of Iowa.

Find the minimum triangle of assigned angles inscribed in a given triangle.

SOLUTION BY ROGER A. JOHNSON, Western Reserve University.

By the well-known theorem of MIQUEL (McClelland, *Geometry of the Circle*, pp. 40–42), if a triangle PQR be inscribed in a given triangle ABC so as to be directly similar, i. e., without being turned over, to a given triangle; then the circles AQR, BRP, CPQ pass through a point O, which, moreover, is fixed for all positions of the inscribed triangle; further, if P, Q, R lie on the sides of the triangle, and not on their extensions, $\angle OPB = \angle OQC = \angle ORA$. If from the point O, lines OP_1, OQ_1, OR_1 be drawn, making equal angles with the sides, their extremities P, Q, R, will form a triangle similar to PQR.

It follows that if the point O be known, we shall obtain the minimum triangle by taking the pedal triangle; that is to say, the triangle whose vertices are the feet of the perpendiculars from O to the sides of ABC.

First inscribe in ABC any triangle of the given species. The following, due to Petersen, is, perhaps, the simplest. Take Y on AC and Z on AB, at random. On YZ construct a triangle XYZ of the given type and in the desired position. Let AX cut BC at P. Through P draw PQ and PR parallel to XY and XZ, respectively. Then QR will be parallel to YZ and PQR will be an inscribed triangle of the desired type.

Now draw the circles AQR, BRP, CPQ, intersecting again at O. From O, drop perpendiculars OL, OM, ON, to the sides. Then LMN is the required triangle.

Also solved by C. N. SCHMALL, FRANK IRWIN, and J. W. CLAWSON.

CALCULUS.

368. Proposed by PAUL CAPRON, Annapolis, Maryland.

Develop $\log_{10} x/\sin x$ and $\log_{10} \tan x/x$ to three terms as functions of $\log_{10} \sec x$, showing that

if x is less than $7° 15'$, then to five decimals, $\log_{10} x = \log_{10} \sin x + \frac{1}{3} \log_{10} \sec x = \log_{10} \tan x - \frac{2}{3} \log_{10} \sec x$.

SOLUTION BY THE PROPOSER.

$$y = \log \frac{x}{\sin x}, \text{ or } e^y = \frac{x}{\sin x}; \quad \frac{dy}{dx} = \frac{1}{x} - \cot x.$$

$$z = \log \sec x, \text{ or } e^z = \sec x; \quad \frac{dz}{dx} = \tan x.$$

Hence,

$$\frac{dy}{dx} = \tan x \frac{dy}{dz}.$$

Also

$$z - y = \log \frac{\tan x}{x}, \text{ or } e^{z-y} = \frac{\tan x}{x}.$$

$$\tan x \frac{dy}{dz} = \frac{1}{x} - \cot x; \quad \tan^2 x \frac{dy}{dz} = \frac{\tan x}{x} - 1, \quad \frac{dy}{dz} = \frac{e^{z-y} - 1}{e^{2z} - 1} = \frac{e^{-y} - e^{-z}}{e^z - e^{-z}}.$$

$$z = 0 = y \quad \text{when} \quad x = 0.$$

$$\left.\frac{dy}{dz}\right]_{z=0} = \frac{0}{0} = \left.\frac{-\frac{dy}{dz}\right] e^{-y} + e^{-z}}{e^z + e^{-z}}\right]_{z=0}; \quad \left.\frac{dy}{dz}\right]_{z=0} = \frac{-\left.\frac{dy}{dz}\right]_{z=0} + 1}{2}; \quad \left.\frac{dy}{dz}\right]_{z=0} = \frac{1}{3}.$$

$$\left.\frac{d^2y}{dz^2}\right]_{z=0} = \frac{1}{(e^z + e^{-z})^2}\left[2 - e^{-y}\left([e^z - e^{-z}]\frac{dy}{dz} + [e^z + e^{-z}]\right)\right]; \quad \left.\frac{d^2y}{dz^2}\right]_{z=0} = \frac{0}{0}.$$

$$\left.\frac{d^3y}{dz^3}\right]_{z=0} = \frac{1}{2(e^{2z} - e^{-2z})}\left[-e^{-y}\left[(e^z - e^{-z})\frac{d^2y}{dz^2} - (e^z - e^{-z})\left(\frac{dy}{dz}\right)^2 + (e^z - e^{-z})\right]\right]_{z=0}$$

$$= \frac{-e^{-y}}{2(e^z + e^{-z})}\left[\frac{d^2y}{dz^2} - \left(\frac{dy}{dz}\right)^2 + 1\right]_{z=0}; \quad \left.\frac{d^3y}{dz^3}\right]_{z=0} = -\frac{8}{45}.$$

$$y = \frac{z}{3} - \frac{8}{45}z^3 \cdots, \quad z - y = \frac{2}{3}z + \frac{8}{45}z^3 \cdots,$$

or

$$\log x - \log \sin x = \frac{1}{3} \log \sec x - \frac{8}{45} (\log \sec x)^3,$$

$$\log_{10} x - \log_{10} \sin x = \frac{1}{3} \log \sec x - \frac{8}{45M} (\log \sec x)^3.$$

Similarly,

$$\log_{10} \tan x - \log_{10} x = \frac{2}{3} \log \sec x + \frac{8}{45M} (\log \sec x)^3,$$

$$\frac{8}{45M} (\log_{10} \sec x)^3 = .000005 \text{ if } \log_{10} \sec x = .0034948, \; x = 7° 15',$$

$$\frac{8}{45M} = .4 \text{ nearly}, \quad = (.40926), \quad = \frac{70}{171} \text{ very closely.}$$

This theorem enables one to dispense without inconvenience with the usual table for small angles. It is well to take $\log_{10}$ c.m. $1' = 6.46372\frac{2}{3} - 10$ in order to account for odd thirds. The theorem gives correct results in most instances for values of θ up to about $10°$. For $\theta = 10°$ there is an error of one unit in the fifth place of decimals.

369. Proposed by I. A. BARNETT, Chicago, Ill.

Compute the definite integral $\int_a^b \log x dx$ by direct summation.

I. Solution by A. M. Harding, University of Arkansas.

Let

$$S_n = dx \cdot \log a + dx \cdot \log (a + dx) + dx \cdot \log (a + 2dx) + \cdots + dx \cdot \log [a + (n-1)dx].$$

Then

$$I = \int_a^b \log x dx = \lim_{n=\infty} S_n, \text{ where } ndx = b - a.$$

Now

$$\log (a + dx) = \log a + \frac{1}{a} dx - \frac{1}{2a^2} dx^2 + \frac{1}{3a^3} dx^3 - \frac{1}{4a^4} dx^4 + \cdots$$

$$\log (a + 2dx) = \log a + \frac{1}{a} 2dx - \frac{1}{2a^2} 2^2 dx^2 + \frac{1}{3a^3} 2^3 dx^3 - \frac{1}{4a^4} 2^4 dx^4 + \cdots$$

$$\log [a + (n-1)dx] = \log a + \frac{1}{a}(n-1)dx - \frac{1}{2a^2}(n-1)^2 dx^2 + \frac{1}{3a^3}(n-1)^3 dx^3 - \frac{1}{4a^4}(n-1)^4 dx^4 + \cdots.$$

Hence,

$$S_n = \log a \cdot ndx + \frac{1}{a}[1 + 2 + 3 + 4 + \cdots + (n-1)]dx^2 - \frac{1}{2a^2}[1^2 + 2^2 + 3^2 + \cdots + (n-1)^2]dx^3$$
$$+ \frac{1}{3a^3}[1^3 + 2^3 + 3^3 + \cdots + (n-1)^3]dx^4 - \frac{1}{4a^4}[1^4 + 2^4 + 3^4 + \cdots + (n-1)^4]dx^5 + \cdots$$
$$= (b-a)\log a + \frac{1}{a}(n^2/2 - n/2)dx^2 - \frac{1}{2a^2}(n^3/3 - n^2/2 + n/6)dx^3$$
$$+ \frac{1}{3a^3}(n^4/4 - n^3/2 + n^2/4)dx^4 - \frac{1}{4a^4}(n^5/5 - n^4/2 + n^3/3 - n/30)dx^5$$
$$+ \frac{1}{5a^5}(n^6/6 - n^5/2 + 5n^4/12 - n^2/12)dx^6 - \cdots;$$

$$\therefore I = \lim S_n = (b-a)\log a + \frac{(b-a)^2}{2a} - \frac{(b-a)^3}{2 \cdot 3a^2} + \frac{(b-a)^4}{3 \cdot 4a^3} - \frac{(b-a)^5}{4 \cdot 5a^4} + \frac{(b-a)^6}{5 \cdot 6a^5} - \cdots$$
$$= (b-a)\log a + \left(1 - \frac{1}{2}\right)\frac{(b-a)^2}{a} - \left(\frac{1}{2} - \frac{1}{3}\right)\frac{(b-a)^3}{a^2} + \left(\frac{1}{3} - \frac{1}{4}\right)\frac{(b-a)^4}{a^3}$$
$$+ \left(\frac{1}{4} - \frac{1}{5}\right)\frac{(b-a)^5}{a^4} - \cdots$$
$$= (b-a)\log a + (b-a)[-(1-b/a) - 1/2(1-b/a)^2 - 1/3(1-b/a)^3 - 1/4(1-b/a)^4 - \cdots]$$
$$+ a\left[-\frac{1}{2}\left(\frac{b-a}{a}\right)^2 + \frac{1}{3}\left(\frac{b-a}{a}\right)^3 - \frac{1}{4}\left(\frac{b-a}{a}\right)^4 + \frac{1}{5}\left(\frac{b-a}{a}\right)^5 - \cdots\right]$$
$$= (b-a)\log a + (b-a)\log\left[1 - \left(1 - \frac{b}{a}\right)\right] + a\left[\log\left\{1 - \left(1 - \frac{b}{a}\right)\right\} + \frac{b-a}{a}\right]$$
$$= (b-a)\log a + (b-a)\log\frac{b}{a} + a\log\frac{b}{a} + b - a$$
$$= (b-a)\log a + b\log\frac{b}{a} + b - a$$
$$= b\log b - a\log a + b - a.$$

II. Solution by the Proposer.

Using the method of division indicated in the diagram, we have,

$$q^n = b \quad \text{and} \quad n = \log\frac{b}{a} - \log q.$$

Hence,

$$\int_a^b \log x\,dx = \lim_{q=1} [a(q-1)\log a + aq(q-1)\log aq + \cdots + aq^{n-1}(q-1)\log aq^{n-1}],$$

$$= \lim_{q=1} a(q-1)[(1+q+\cdots+q^{n-1})\log a + (q+2q^2+\cdots+(n-1)q^{n-1})\log q],$$

$$= \lim_{q=1} a(q-1)\left[\left(\frac{q^n-1}{q-1}\right)\log a + \left\{\frac{q-nq^n+(n-1)q^{n+1}}{(1-q)^2}\right\}\log q\right],*$$

$$= \lim_{q=1} a\left[(q^n-1)\log a + \frac{q(1-q^n)}{q-1}\log q + nq^n\log q\right],$$

$$= \lim_{q=1} a\left[\left(\frac{b-a}{a}\right)\log a + \frac{q}{q-1}\left(\frac{a-b}{a}\right)\log q + \frac{b}{a}\log\frac{b}{a}\right],$$

$$= b\log b - a\log a + b - a.$$

Also solved similarly by P. PENALVER.

MECHANICS.

292. Proposed by C. N. SCHMALL, New York City.

In a bombardment, a battleship directs its fire at a fort standing on a hill whose height is a feet above sea level. The angle of elevation of the fort is found to be ϕ. If the initial velocity of the projectile is v, show that the fort will *not* be struck if $v < \sqrt{ag(1+\csc\phi)}$.

SOLUTION BY PAUL CAPRON, Annapolis, Maryland.

If the projectile is fired at an elevation θ, its trajectory may be represented by $x = vt\cos\theta$, $y = vt\sin\theta - \frac{1}{2}gt^2$, or by

$$y = x\tan\theta - \frac{gx^2}{2v^2}\sec^2\theta.$$

To find the envelope of all such trajectories, we have

$$0 = x\sec^2\theta - \frac{gx^2}{v^2}\sec^2\theta\tan\theta, \qquad \text{or} \qquad \tan\theta = \frac{v^2}{gx};$$

whence

$$y = \frac{v^2}{2g} - \frac{gx^2}{2v^2}$$

is the envelope.

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* To sum

$$q + 2q^2 + 3q^3 + \cdots + (n-1)q^{n-1} = q(1 + 2q + 3q^2 + \cdots + (n-1)q^{n-2}) = q\Sigma$$

$$\Sigma = 1 + 2q + 3q^2 + \cdots + (n-1)q^{n-2}$$

$$-q\Sigma = -q - 2q^2 - \cdots - (n-2)q^{n-2} - (n-1)q^{n-1}$$

$$\therefore (1-q)\Sigma = 1 + q + q^2 + \cdots + q^{n-2} - (n-1)q^{n-1}$$

$$= \frac{q^{n-1}-1}{q-1} - (n-1)q^{n-1}$$

$$\therefore q\Sigma = \frac{q - nq^n + (n-1)q^{n+1}}{(1-q)^2}.$$

The condition that the fort shall not be hit is given by

$$x = a\cot\phi, \qquad a > y;$$

or by

$$a > \frac{v^2}{2g} - \frac{ga^2\cot^2\phi}{2v^2} \qquad \text{or} \qquad v^4 - 2agv^2 < a^2g^2\cot^2\phi,$$

or

$$(v^2 - ag)^2 < a^2g^2\csc^2\phi, \qquad \text{or} \qquad v < \sqrt{ag(1+\csc\phi)}.$$

Also solved by L. SIVIAN, F. C. FEEMSTER, A. M. HARDING, and CLIFFORD N. MILLS.

NUMBER THEORY.

220. Proposed by E. T. BELL, Seattle, Washington.

Let $[m/n]$ denote the greatest integer that is not greater than m/n; and let the two sets,

$$\left[\frac{m}{m-1}\right]; \left[\frac{m+1}{m-2}\right]; \left[\frac{m+2}{m-3}\right]; \cdots \left[\frac{2m-3}{2}\right],$$

$$\left[\frac{m-1}{m-1}\right]; \left[\frac{m}{m-2}\right]; \left[\frac{m+1}{m-3}\right]; \cdots \left[\frac{2m-4}{2}\right],$$

be denoted by (A) and (B) respectively.

Prove that a necessary and sufficient condition that $2m-1$ be a prime number is that the excess of the number of even integers in (A) over the number of even integers in (B) shall be equal to the excess of the number of odd integers in (A) over the number of odd integers in (B).

SOLUTION BY ELIJAH SWIFT, University of Vermont.

Unless one of the quantities $m/(m-1)$, $(m+1)/(m-2)$, $\cdots$, $(2m-3)/2$ is an integer the two sets (A) and (B) will be exactly the same. If one of the above quantities is an integer we shall show (1) that it is an even integer; and (2) that $2m-1$ will then possess a factor.

Let $(m+k)/(m-k-1) =$ an integer, say α. This is equivalent to the equation $2m-1 = (\alpha+1)(m-k-1)$, which proves the statements above.

Suppose now that $2m-1$ is prime. Then sets (A) and (B) will be exactly the same. Hence the condition is necessary. If $2m-1$ is composite, each factor is odd. Call them $\alpha+1$ and $m-k-1$, k must be as large as 1. But $2m-1 = (\alpha+1)(m-k-1)$ is the same as $(m+k)/(m-k-1) = \alpha =$ an integer. Hence, at least one of the integers of (A) is even and the condition can not be fulfilled. Hence the condition is sufficient.

QUESTIONS AND DISCUSSIONS.

EDITED BY U. G. MITCHELL, University of Kansas.

At the time of making up copy for this issue replies had not yet been received for numbers 4, 8, 12, 13, 16, 20 and further replies are desired for numbers 23, 24 and 25. May we have these for the September issue?

REPLIES.

11. In our courses of study is it desirable to give more consideration to vector analysis? What topics should be included in a first treatment of this subject?

REPLY BY E. L. REES, University of Kentucky.

The writer is much disappointed that question eleven has not elicited sufficient interest to bring forth a full discussion, such, for example, as that on question three. While he feels that synthetic geometry, as outlined by Professor Bussey, makes a very desirable undergraduate course, he also feels that, among the somewhat large number of other possible courses, vector analysis is fully as practicable and valuable as synthetic geometry.

The following, in the opinion of the writer, are some of the merits of such a course:

(*a*) The student's interest in mathematics is stimulated. He is charmed by the symmetry and beauty of the new notation, and, by frequent translations from the vector language to the scalar, he is constantly reminded of the greater brevity and power of the new symbols.

(*b*) The esthetic element, which is so much in evidence in the higher courses of mathematics and which is too often obscured in the grind of the elementary work, impresses itself easily on the student's mind.

(*c*) The student is made to feel the power and comprehensiveness of a new mathematical instrument in the vector methods, without being overburdened and discouraged by the difficulties of the subject.

(*d*) The very wide range of applications serves at once as a review of a large part of his college work and as a means of introducing certain new subjects. This ability to treat such a large variety of subjects, already familiar to the student, by one uniform method, and to introduce entirely new subjects by this method, gives him a more elevated point of view and tends to broaden his mental vision.

(*e*) The student is relieved of the monotony of his other algebraic work by the freshness of a new symbolism with its many interesting interpretations.

(*f*) It affords a most attractive introduction to the theory of space curves and surfaces.

(*g*) Finally, the difficulties of the subject are not too great for students who elect mathematics as their undergraduate major, if only the algebra and a brief introduction to the calculus of a vector function of a scalar be given.

The writer has taught this subject to juniors at the University of Kentucky for a number of years with very satisfactory results. The course has been given by lectures with collateral reading in Gibbs-Wilson and Coffin.

It is hoped that this imperfect discussion will open the way for a much fuller discussion, either by way of criticism or enlargement on what has been written.

25. In an investigation in physics Mr. Mason E. Hufford, 525 S. Park Ave., Bloomington, Indiana, has need of the values of the Bessel functions $J_0(x)$ and $J_1(x)$ for positive real values of x up to $x = 100$. Have tables been constructed to this extent? What is the most ready means by which the desired values may be computed to any required degree of accuracy?

REPLY BY S. A. COREY, Hiteman, Iowa.

Values of Bessel functions $J_0(x)$ and $J_1(x)$ may, for small values of x, be obtained by ordinary methods of quadrature from the definite integrals

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\phi)\,d\phi,$$

$$J_1(x) = \frac{x}{\pi}\int_0^{\pi} \sin^2\phi\cos(x\cos\phi)\,d\phi,$$

by taking Δx equal to $\pi/9$, $\pi/12$, $\pi/18$, or for greater accuracy still smaller values.

For larger values of x use the asymptotic series

$$J_0(x) = \sqrt{\frac{2}{\pi x}}\left[\cos\left(\frac{\pi}{4}-x\right)\left\{1-\frac{1^2\cdot 3^2}{1\cdot 2(8x)^2}+\frac{1^2\cdot 2^2\cdot 3^2}{1\cdot 2\cdot 3\cdot 4(8x)^4}\cdots\right\}\right.$$
$$\left.-\sin\left(\frac{\pi}{4}-x\right)\left\{\frac{1^2}{1\cdot 8x}-\frac{1^2\cdot 3^2\cdot 5^2}{1\cdot 2\cdot 3(8x)^3}+\cdots\right\}\right],$$

$$J_1(x) = \sqrt{\frac{2}{\pi x}}\left[\left\{1-\frac{(4-1)^2(4-3)^2}{1\cdot 2(8x)^2}+\frac{(4-1)^2(4-3)^2(4-5)^2(4-7)^2}{1\cdot 2\cdot 3\cdot 4(8x)^4}-\cdots\right\}\right.$$
$$\left.\times\cos\left(\frac{3\pi}{4}+x\right)-\left\{\frac{4}{1\cdot 8x}+\frac{(4-1)^2(4-3)^2(4-5)^2}{1\cdot 2\cdot 3(8x)^3}+\cdots\right\}\sin\left(\frac{3\pi}{4}-x\right)\right].$$

These asymptotic series are of such a nature that if we stop the calculations with any term the resulting error is less than the next term. As x increases these series become rapidly convergent, so that any degree of accuracy required in any physical investigation is readily obtainable.

NOTES AND NEWS.

EDITED BY W. D. CAIRNS.

"The mathematical analysis of electrical and optical wave-motion" is the title of a book by Dr. HARRY BATEMAN just published by G. P. Putnam's Sons.

Professor H. S. WHITE of Vassar College gave an address on March 5 under the auspices of the Syracuse University chapter of Sigma, his subject being "Mathematics in nineteenth century science."

In *Science* for April 2 President R. S. WOODWARD develops further the differences between his treatment of the deviations of freely falling bodies and the methods of Professor F. R. MOULTON and Professor W. H. ROEVER. (See this MONTHLY of December, 1913, and September, 1914.)

Professor C. W. MCCORD, who retired from active service at the Stevens Institute of Technology in 1906, where he had been professor of mechanical drawing since 1871, died in Hoboken, N. J., on April 13, 1915. He was the author of a book on descriptive geometry and was justly honored as the draughtsman who with Captain John Ericsson in 1859 made the plans for the *Monitor*.

Professor R. D. CARMICHAEL, of Indiana University, and Professor ARNOLD DRESDEN, of the University of Wisconsin, will be members of the staff in mathematics at the University of Chicago during the summer quarter, 1915. Professors L. E. DICKSON and H. E. SLAUGHT will be absent on vacation.

DR. DANIEL BUCHANAN, of Queen's University, Kingston, Ontario, has been promoted to an associate professorship in mathematics and has been appointed director of the college observatory.

Professor C. J. KEYSER, of Columbia University, and H. W. TYLER, of the Massachusetts Institute of Technology, will lecture this summer at the University of California, the former on "History and significance of central mathematical ideas and doctrines," the latter on "History of mathematical science."

PROFESSOR H. S. WHITE, of Vassar College, was elected a member of the National Academy of Sciences at the annual meeting held last April in Washington, D. C. The other members representing pure mathematics in this Academy are Professors BÔCHER, BOLZA, DICKSON, MOORE, OSGOOD, STOREY, and VAN VLECK.

MR. GEORGE RUTLEDGE, of the University of Illinois, has accepted an instructorship in mathematics in the Massachusetts Institute of Technology.

At the University of Kansas, Dr. U. G. MITCHELL has been promoted to an associate professorship, and C. H. ASHTON to a professorship in the department of mathematics. Dr. E. B. STOUFFER, who went there last year from the University of Illinois, is an assistant professor in the department.

An article on "The place of the textbook in mathematical teaching" by Mr. EDMUND LIGHTLEY appeared in the *School World* for January.

Professor de Morgan's "Budget of Paradoxes," long since out of print, has been edited by Professor DAVID EUGENE SMITH with extended biographical, historical and explanatory notes. It will appear soon from the press of the Open Court Publishing Company.

The cover for the May issue was dated "April" by an oversight of the printer. The printed slip herewith may be used to correct the error. The next issue of the MONTHLY, will be for September, 1915.

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HISTORY OF ZENO'S ARGUMENTS ON MOTION:

PHASES IN THE DEVELOPMENT OF THE THEORY OF LIMITS.

By FLORIAN CAJORI, Colorado College.

VIII.

D. VIEWED IN THE LIGHT OF AN IDEALISTIC CONTINUUM.

The full explanation of Zeno's paradoxes requires two ideas which are very familiar to the modern mathematician, namely, the acceptance of the existence of actually infinite aggregates and the idea of a connected and perfect continuum. The first concept, that of actual infinity, as opposed to potential infinity, had been under contemplation ever since the time of Aristotle. Men like St. Augustine, Galileo, Pascal, Volder, Schultze seemed to have had a more or less non-contradictory conception of it. Others only denied it. Among the latter were both philosophers and mathematicians; the list includes men like Thomas Aquinas, Gerdil, Descartes, Spinoza, Leibniz, Locke, Lotze, Renouvier, Moigno, Cauchy, Gauss, and of course many others.[1] Wonderful insight into this matter was possessed by Galileo. He showed for instance that there were as many integers that were perfect squares as there were integers altogether. Strange to say, Galileo's argument has been misinterpreted by some recent writers. Instead of accepting the conclusion as legitimate for that sort of infinity, as it was accepted by Galileo, these writers declared the conclusion absurd, hence the hypothesis of actual infinity a myth. Among men taking this view was the French philosopher F. Pillon.[2] In 1831 (July 12) the great K. F. Gauss of Göttingen wrote a letter to Schumacher in which he declares himself as opposed to the actual infinity in mathematics.

[1] For details see Georg Cantor, "Ueber die verschiedenen Standpunkte in Bezug auf das actuele Unendliche" in *Zeitschr. f. Philos. u. philos. Kritik*, Bd. 88, p. 224.

[2] *L'année philosophique*, I, 1890, p. 84, quoting among others Cauchy's *Sept leçons de physique générale*, 1868, as follows: "Cette proposition fondamentale, démontrée par Galilée (qu'on ne saurait admettre une suite ou série actuellement composée d'un nombre infini de termes), s'applique aussi bien à une série de termes ou d'objects, qui ont existé." Cauchy was strongly influenced on this matter by the writings of Gerdil.